1 Complex Numbers and Some Basic Algebraic Manipulations

Definition 1.1. *By introducing the pure imaginary number i satisfying $i^2 = -1$, the set of complex numbers $\mathbb{C}$ is defined by*

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\},$$

where $\mathbb{R}$ is the set of real numbers.

Definition 1.2. *For a complex number $z = x + yi$, $x, y \in \mathbb{R}$, x and y are the real and imaginary parts of z. We denote*

$$\operatorname{Re} z = x \quad \textit{and} \quad \operatorname{Im} z = y.$$

If $\operatorname{Im} z = 0$, then z is a real number. If $\operatorname{Re} z = 0$, z is called a pure imaginary number. Two complex numbers z_1 and z_2 are equal if

$$\operatorname{Re} z_1 = \operatorname{Re} z_2 \quad \textit{and} \quad \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

Remark 1.3. *For $a, b \in \mathbb{R}$, one of the following three relations holds: (i) $a < b$; (ii) $a = b$; (iii) $a > b$. But for complex numbers z_1 and z_2, we do not have $z_1 > z_2$ or $z_1 < z_2$.*

Definition 1.4 (Addition)**.** *For $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we define the sum $z_1 + z_2$ to be*

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i.$$

Property 1.5.

(i) (Commutative law) $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$.

(ii) (Associative law) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

(iii) (Summation identity) There is $0 \in \mathbb{C}$ such that $z + 0 = z$ for all $z \in \mathbb{C}$.

(iv) (Summation inverse) For all $z \in \mathbb{C}$, there is $-z \in \mathbb{C}$ such that $z + (-z) = 0$.

Remark 1.6.

(i) $0 = 0 + 0i$.

(ii) For $z = x + yi$, $x, y \in \mathbb{R}$, $-z = (-x) + (-y)i$.

Definition 1.7 (Subtraction)**.** *For $z_1, z_2 \in \mathbb{C}$, we define the subtraction $z_1 - z_2$ to be*

$$z_1 - z_2 = z_1 + (-z_2).$$

Formal Calculation. *Assuming that the commutative law, associative law and the distributive law hold for complex numbers, for $x_1, x_2, y_1, y_2 \in \mathbb{R}$,*

$$\begin{aligned}(x_1 + y_1 i)(x_2 + y_2 i) &= x_1x_2 + x_1y_2 i + x_2y_1 i + y_1y_2 i^2 \\ &= x_1x_2 + x_1y_2 i + x_2y_1 i - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i.\end{aligned}$$

Definition 1.8 (Multiplication)**.** *For $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we define the product $z_1 z_2$ to be*

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i.$$

Property 1.9.

(i) (Commutative law) $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in \mathbb{C}$.

(ii) (Associative law) $z_1(z_2 z_3) = (z_1 z_2)z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

(iii) (Distributive law) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

(iv) (Multiplication identity) There is $1 \in \mathbb{C}$ such that $z \cdot 1 = z$ for all $z \in \mathbb{C}$.

(v) (Multiplication inverse) For all $z \in \mathbb{C}\backslash\{0\}$, there is $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$.

Remark 1.10.

(i) If $z_1, z_2 \in \mathbb{C}$, $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$, or possibly $z_1 = z_2 = 0$.

(ii) $1 = 1 + 0i$.

(iii) For $z = x + yi$, $x, y \in \mathbb{R}$, $z^{-1} = \dfrac{x}{x^2+y^2} + \dfrac{-y}{x^2+y^2} i$.

(iv) Sometimes, we denote z^{-1} by $\dfrac{1}{z}$.

(v) For $z \in \mathbb{C}\backslash\{0\}$, $n \in \mathbb{N}$, z^n is defined inductively by

$$\begin{cases} z^k = z^{k-1} z & \text{for } k \in \mathbb{N}, \\ z^0 = 1. \end{cases}$$

(vi) (Binomial formula) For $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, $n \in \mathbb{N}$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Definition 1.11 (Division)**.** *For $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$, we define the division by*

$$\frac{z_1}{z_2} = z_1 z_2^{-1}.$$

Remark 1.12. *For $z_1, ..., z_4 \in \mathbb{C}$, $z_3 \neq 0$, $z_4 \neq 0$,*

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4}.$$

Example 1.13.

$$\frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2+3i)(2-3i)} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

Definition 1.14 (Euler's formula)**.** *For $y \in \mathbb{R}$,*

$$e^{yi} = \cos y + i \sin y.$$

Formal Calculation. *Recall that the exponential function for real numbers admits a Taylor expansion. For $x \in \mathbb{R}$,*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If the above expansion holds for complex numbers, particularly for pure imaginary numbers, we have

$$e^{yi} = \sum_{n=0}^{\infty} \frac{y^n i^n}{n!}.$$

Since $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, and $i^{4k+3} = -i$, for all $k \in \mathbb{N} \cup \{0\}$, we can divide the above series into four parts as follows.

$$\begin{aligned} e^{yi} &= \sum_{k=0}^{\infty} \frac{y^{4k} i^{4k}}{(4k)!} + \sum_{k=0}^{\infty} \frac{y^{4k+1} i^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{y^{4k+2} i^{4k+2}}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{y^{4k+3} i^{4k+3}}{(4k+3)!} \\ &= \sum_{k=0}^{\infty} \frac{y^{4k}}{(4k)!} + i \sum_{k=0}^{\infty} \frac{y^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{y^{4k+2}}{(4k+2)!} - i \sum_{k=0}^{\infty} \frac{y^{4k+3}}{(4k+3)!}. \end{aligned}$$

Combining the real parts and the imaginary parts together, it follows

$$\begin{aligned} e^{yi} &= \left(\sum_{k=0}^{\infty} \frac{y^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{y^{4k+2}}{(4k+2)!} \right) + i \left(\sum_{k=0}^{\infty} \frac{y^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{y^{4k+3}}{(4k+3)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \\ &= \cos y + i \sin y. \end{aligned}$$

Definition 1.15. *For $z = x + yi$, $x, y \in \mathbb{R}$,*

$$e^z = e^x(\cos y + i \sin y).$$

Proposition 1.16. *For $z_1, z_2 \in \mathbb{C}$,*

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}$$

Remark 1.17. *For $z \in \mathbb{C}$, the complex exponential function also has the Taylor expansion*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

2 Geometric Representation of Complex Number Field

A complex number $z = x + yi$, $x, y \in \mathbb{R}$, can be identified as a point (x, y) in $\mathbb{R}^2$. We can interpret the algebraic manipulations of complex numbers in the following geometric way.

Addition. *Given $z_1, z_2 \in \mathbb{C}$, we can construct a parallelogram with edges $\overline{0z_1}$ and $\overline{0z_2}$. Then the fourth vertex, different from 0, z_1 and z_2, corresponds to $z_1 + z_2$.*

Subtraction. *$z_1 - z_2$ denotes the vector starting from z_2 and ending at z_1.*

Polar Coordinates. *For $(x, y) \in \mathbb{R}^2$, we have the polar coordinates*

$$(x, y) = (\rho \cos\theta, \rho \sin\theta),$$

where $\rho = \sqrt{x^2 + y^2}$, $\theta \in \mathbb{R}$. The corresponding complex number $z = x + yi$ can be represented as

$$z = x + yi = \rho \cos\theta + i\rho \sin\theta = \rho(\cos\theta + i \sin\theta).$$

By using Euler's formula, $\cos\theta + i\sin\theta = e^{i\theta}$, we obtain

$$z = \rho e^{i\theta}.$$

Definition 2.1. *For $z = x + yi = \rho e^{i\theta} \in \mathbb{C}$, $\rho = \sqrt{x^2 + y^2}$ is called the modulus of z, denoted by $|z|$. That is, the modulus of z is*

$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

And for $z \neq 0$, we call θ an argument of z and define $\arg z$ to be the set of all argument of z.

Example 2.2.

(i) $|-3 + 2i| = \sqrt{13}$.

(ii) $|1 + 4i| = \sqrt{17}$.

Remark 2.3.

(i) Geometrically, $|z|$ is the distance between (x, y) and the origin.

(ii) $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$ *and* $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$.

(iii) For $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1||z_2|$. And $|z^{-1}| = |z|^{-1}$ if $z \neq 0$.

(iv) $|z^n| = |z|^n$ for $z \in \mathbb{C}$, $n \in \mathbb{N}$.

(v) For $z = 0$, θ is undefined.

(vi) For $z \neq 0$, θ is defined up to $2k\pi$, $k \in \mathbb{Z}$. If we restrict θ to be a number in $(-\pi, \pi]$, then the argument for a complex number can be uniquely determined. That is, there is a unique $\Theta \in (-\pi, \pi]$ such that $\Theta \in \arg z$. We call Θ the principal argument of z, denoted by $\operatorname{Arg} z$.

(vii) For $z \neq 0$,

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k \in \mathbb{Z}\}.$$

Example 2.4.

$$\operatorname{Arg}(-1-i) = -\frac{3\pi}{4}.$$
$$\arg(-1-i) = \left\{-\frac{3\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}.$$

Proposition 2.5 (Triangle inequality)**.** *For $z_1, z_2 \in \mathbb{C}$,*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. For the second inequality, we can construct a triangle with vertices 0, z_1 and $z_1 + z_2$. Then length of the edge between 0 and $z_1 + z_2$ if bounded by the sum of the length of the other two. The inequality then follows. As for the first inequality, we can apply the inequality we just proved to get

$$|z_1| = |(z_1 + z_2) + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|.$$

That is,

$$|z_1| - |z_2| \leq |z_1 + z_2|.$$

Interchanging the roles of z_1 and z_2, we obtain

$$|z_2| - |z_1| \leq |z_1 + z_2|.$$

The last two inequalities complete the proof. □

Proposition 2.6. *For $z_1, ..., z_n \in \mathbb{C}$,*

$$|z_1 + ... + z_n| \leq |z_1| + ... + |z_n|.$$

Proof. By mathematical induction. □

Example 2.7. *We can use the triangle inequality to estimate $3 + z + z^2$ for all z with $|z| = 2$ as follows. By the triangle inequality,*

$$|3 + z + z^2| \leq 3 + |z| + |z^2|.$$

Since $|z^2| = |z|^2$ and $|z| = 2$, the above estimate is reduced to

$$|3 + z + z^2| \leq 3 + |z| + |z^2| = 3 + |z| + |z|^2 = 9.$$

Multiplication. *For two complex numbers $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$, we have*

$$z_1 z_2 = \rho_1 \rho_2 e^{i(\theta_1 + \theta_2)}.$$

That is,

$$|z_1 z_2| = \rho_1 \rho_2$$

and

$$\arg(z_1 z_2) = \{\theta_1 + \theta_2 + 2k\pi : k \in \mathbb{Z}\}.$$

Remark 2.8. *When a complex number* $z_1 = \rho_1 e^{i\theta_1}$ *is multiplied by another complex number* $z_2 = \rho_2 e^{i\theta_2}$*, we have the modulus of the product* $|z_1 z_2| = \rho_2 |z_1|$*. That is, it corresponds to stretch or compress the vector* z_1*. Since* $\theta_1 + \theta_2$ *is an argument of* $z_1 z_2$*, the direction of* $z_1 z_2$ *can be obtained by rotating the direction of* z_1 *counterclockwise by* θ_2 *if* $\theta_2 > 0$*, or clockwise by* $-\theta_2$ *if* $\theta_2 < 0$*.*

Remark 2.9.

(i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ *in the sense of set addition. But in general, the equality*

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

is false.

(ii) For complex number $z = \rho e^{i\theta}$, $\rho > 0$, $z^{-1} = \rho^{-1} e^{-i\theta}$*.*

(iii) For $z \neq 0$, $\arg(z^{-1}) = -\arg z$*.*

(iv) For $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$*.*

(v) For complex number $z = \rho e^{i\theta}$, $\rho > 0$, $z^n = \rho^n e^{in\theta}$ *for all* $n \in \mathbb{Z}$*.*

(vi) (de Moivre's formula) By using (v) with $\rho = 1$*, for* $n \in \mathbb{Z}$*, we have*

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

That is,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{2.1}$$

Example 2.10. *If* $z_1 = -1$ *and* $z_2 = i$*, then*

$$\operatorname{Arg} z_1 = \pi \quad \textit{and} \quad \operatorname{Arg} z_2 = \frac{\pi}{2}.$$

However,

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \neq \frac{3\pi}{2} = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

Example 2.11. *In order to find the principal argument of* $z = \dfrac{i}{-1-i}$*, we start by writing*

$$\arg z = \arg i - \arg(-1-i).$$

Since

$$\operatorname{Arg} i = \frac{\pi}{2} \quad \textit{and} \quad \operatorname{Arg}(-1-i) = -\frac{3\pi}{4},$$

we have that $\dfrac{5\pi}{4} \in \arg z$*. Therefore,*

$$\operatorname{Arg} z = -\frac{3\pi}{4}.$$

Example 2.12. *By (2.1) with $n = 2$, we have*

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$$

That is,

$$\left(\cos^2\theta - \sin^2\theta\right) + i\,(2\sin\theta\cos\theta) = \cos 2\theta + i\sin 2\theta.$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \text{and} \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

3 Some Basic Geometric Objects Represented in Complex Theory

Example 3.1 (Circles)**.** *A circle with center z_0 and radius r_0 is given by $\{z \in \mathbb{C} : |z - z_0| = r_0\}$.*

Example 3.2. *The interior part of the circle given in Example 3.1 is the set $\{z \in \mathbb{C} : |z - z_0| < r_0\}$.*

Example 3.3. *The exterior part of the circle given in Example 3.1 is the set $\{z \in \mathbb{C} : |z - z_0| > r_0\}$.*

Example 3.4 (Ellipses)**.** *An ellipse with foci z_1 and z_2 is given by $\{z \in \mathbb{C} : |z - z_1| + |z - z_2| = d\}$. Here d is the length of the long axis.*

Example 3.5 (Lines)**.** *Given two complex numbers z_1 and z_2, they determine a straight line L such that L passes across z_1 and z_2. For all points on L, denoted by z, the direction from z_1 to z_2 and the direction from z_1 to z are either the same or different by π. Therefore, by polar coordinates, if $z_2 - z_1 = \rho e^{i\theta}$, then it must hold*

$$z - z_1 = re^{i\theta} \quad \text{or} \quad z - z_1 = re^{i(\theta+\pi)}.$$

Here ρ and r are moduli of $z_2 - z_1$ and $z - z_1$, respectively. Therefore, we have

$$\text{either} \quad \frac{z - z_1}{z_2 - z_1} = \frac{r}{\rho} \quad \text{or} \quad \frac{z - z_1}{z_2 - z_1} = -\frac{r}{\rho}.$$

In either case, $\dfrac{z - z_1}{z_2 - z_1}$ is real, provided that z lies on the line L. The converse is also true. So in the complex theory, line L determined by z_1 and z_2 can be represented by

$$L = \left\{ z \in \mathbb{C} : \operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \right\}. \tag{3.1}$$

Example 3.6. *Find all points which satisfy*

$$\operatorname{Im}\left(\frac{z + 1 - 3i}{4 - i}\right) = 0.$$

The condition given in this example is quite similar to (3.1). It is a particular case of (3.1) when we have

$$-z_1 = 1 - 3i \quad \text{and} \quad z_2 - z_1 = 4 - i.$$

That is, $z_1 = -1 + 3i$ and $z_2 = 3 + 2i$. By the discussion in Example 3.5, the points in this example represent a line passing across $-1 + 3i$ and $3 + 2i$.

Example 3.7 (Another representation for circles)**.** *A circle can be uniquely determined by given three points which are not on the same line. Suppose that C is the circle passing across z_1, z_2 and z_3. For another point z on C, without loss of generality, we assume that z_1, z_2, z_3 and z are clockwise distributed. Other cases can be similarly considered. Then by fundamental geometry, it holds*

$$\angle z_1 z_3 z_2 = \angle z_1 z z_2.$$

The reason is that these two angles correspond to the same arc on the circle C. Notice that we can rotate the vector $z_3 - z_2$ counterclockwise by the angle $\angle z_1 z_3 z_2$, the resulted vector must have the same direction as $z_3 - z_1$. Therefore, we have

$$z_3 - z_1 = \lambda_1 (z_3 - z_2) e^{i \angle z_1 z_3 z_2}$$

for some $\lambda_1 > 0$. Similarly, we have

$$z - z_1 = \lambda_2 (z - z_2) e^{i \angle z_1 z z_2}$$

for some $\lambda_2 > 0$. Here λ_1 and λ_2 are positive real numbers. Since $\angle z_1 z_3 z_2 = \angle z_1 z z_2$, the last two equalities yield

$$\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \frac{\lambda_2}{\lambda_1}.$$

This furthermore implies

$$\operatorname{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0.$$

One can apply similar arguments above for the other possible positions of z on C. The last equality always holds once z is on C. Therefore, we conclude that

$$C = \left\{ z \in \mathbb{C} : \operatorname{Im}\left[\left(\frac{z - z_1}{z - z_2}\right) \Big/ \left(\frac{z_3 - z_1}{z_3 - z_2}\right)\right] = 0 \right\}. \tag{3.2}$$

Example 3.8. *Find all points which satisfy*

$$\operatorname{Im}\left(\frac{1}{z}\right) = 1.$$

Notice that

$$\operatorname{Im}\left(\frac{1}{z}\right) = 1 = \operatorname{Im}(i).$$

Therefore,

$$0 = \operatorname{Im}\left(\frac{1}{z} - i\right) = \operatorname{Im}\left(\frac{1 - iz}{z}\right) = \operatorname{Im}\left(\frac{z + i}{z} \cdot (-i)\right).$$

Compare with (3.2), we have in this example

$$-z_1 = i, \quad z_2 = 0, \quad \textit{and} \quad \frac{z_3 - z_1}{z_3 - z_2} = -i.$$

Equivalently, it holds $z_1 = -i$, $z_2 = 0$, $z_3 = \frac{1}{2} - \frac{i}{2}$. *It represents a circle passing across these three points. Analytically all points in this example satisfy*

$$\left| z + \frac{i}{2} \right| = \frac{1}{2}.$$

Example 3.9 (Side of a line)**.** *Given different* z_1 *and* z_2 *in* $\mathbb{C}$*, we can determine a line* L*. There are two directions if a line is given. One direction is from* z_1 *to* z_2*, while another direction is from* z_2 *to* z_1*. The concept of side is related to the direction that we are using. If we fix a direction by starting from* z_1 *to* z_2*, then all points on the left form the left-hand side of the line* L*, while all points on the right form the right-hand side of the line* L*. Pay attention that the left-hand side and the right-hand side depend on the direction that we are using. Suppose that the direction is given by starting from* z_1 *to* z_2*. Then, for an arbitrary point* z *on the left-hand side, we can rotate* $z_2 - z_1$ *counterclockwise by an angle* $\theta_0 \in (0, \pi)$ *to the direction given by* $z - z_1$*. In other words,*

$$z - z_1 = \lambda_0 (z_2 - z_1) e^{i\theta_0},$$

for some $\lambda_0 > 0$ *and* $\theta_0 \in (0, \pi)$*. From the above equality, we have*

$$\operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1} \right) = \lambda_0 \sin \theta_0 > 0.$$

Similarly, if z *is on the right-hand side of* L *with the direction given by pointing from* z_1 *to* z_2*, then it holds*

$$\operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1} \right) = \lambda_0 \sin \theta_0 < 0.$$

The above arguments and (3.1) implies that given z_1 *and* z_2*, all points satisfy (3.1) must lie on the line across* z_1 *and* z_2*. If*

$$\operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1} \right) > 0,$$

then z *lies on the left-hand side of* L *with the direction from* z_1 *to* z_2*. If*

$$\operatorname{Im}\left(\frac{z - z_1}{z_2 - z_1} \right) < 0,$$

then z *lies on the right-hand side of* L*.*

Example 3.10. *Find all points satisfying*

$$\operatorname{Im}\left(\frac{z + 1 - 3i}{4 - i} \right) > 0. \tag{3.3}$$

By example 3.6, points satisfy

$$\operatorname{Im}\left(\frac{z + 1 - 3i}{4 - i} \right) = 0$$

lie on the line L *across* $z_1 = -1 + 3i$, $z_2 = 3 + 2i$*. By Example 3.9,* z *satisfying (3.3) must be on the left-hand side of* L *with the direction from* z_1 *to* z_2*.*

Example 3.11 (Reflection in the real axis)**.** *In complex theory, given a complex number $z = x + yi$, we have an operator to find its symmetric point with respect to the x-axis. In fact, the symmetric point of (x, y) with respect to the x-axis is $(x, -y)$. This symmetric point corresponds to the number $x - yi$. In the future, we denote by $\overline{z} = x - yi$ the symmetric point of z with respect to the x-axis.*

Definition 3.12 (Complex conjugates)**.** *For $z = x + yi \in \mathbb{C}$, the symmetric point of z with respect to the real axis, i.e.,*

$$\overline{z} = x - yi,$$

is called the conjugate of z.

Property 3.13.

(i) $\overline{\overline{z}} = z$ *and* $|\overline{z}| = |z|$.

(ii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, *and* $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. *If* $z_2 \neq 0$, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$.

(iii) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ *and* $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$.

(iv) $z\overline{z} = |z|^2$.

Example 3.14 (Computation of roots)**.** *Given $z = \rho e^{i\theta}$, we can easily calculate $z^n = \rho^n e^{in\theta}$. Conversely, if we are given $a = \rho_0 e^{i\theta_0} \neq 0$, we can also find z such that $z^n = a$, $n \in \mathbb{N}$. Indeed, suppose that $z = \rho e^{i\theta}$, then $z^n = a$ can be equivalently written as*

$$\rho^n e^{in\theta} = \rho_0 e^{i\theta_0}.$$

It then follows

$$\rho = \rho_0^{1/n} \quad \text{and} \quad e^{i\left(n\theta - \theta_0\right)} = 1.$$

ρ is uniquely determined. But cosine and sine are periodic function, the second equality above can only imply

$$n\theta - \theta_0 = 2k\pi,$$

for some $k \in \mathbb{Z}$. Therefore, θ is not uniquely determined. All z with $\rho = \rho_0^{1/n}$ and θ given by

$$\frac{\theta_0}{n} + \frac{2k\pi}{n}, \quad k \in \mathbb{Z},$$

will satisfy the equation $z^n = a$. Such z is called an n-th root of a. Notice that we can only have n different roots for a given non-zero complex number a.

Definition 3.15. *For $z \in \mathbb{C}$, $n \in \mathbb{N}$, we denote $z^{1/n}$ the set of n-th roots of z. If $z = \rho e^{i\theta} \neq 0$,*

$$z^{1/n} = \left\{ \sqrt[n]{\rho} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} : k = 0, 1, ..., n-1 \right\}.$$

In particular, if $z = \rho e^{i\theta} \neq 0$ with $\theta \in (-\pi, \pi]$, i.e., $\theta = \operatorname{Arg} z$, then

$$\sqrt[n]{\rho} e^{i\theta/n} = \sqrt[n]{\rho} e^{i\mathrm{Arg} z/n}$$

is called the principal n-th root of z.

Remark 3.16. *If $z = 0$, all the n-th roots are 0.*

Example 3.17. *To find all of the fourth roots of -16, we have*

$$-16 = 16e^{i\pi}.$$

Therefore,

$$(-16)^{1/4} = \left\{2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4}\right\}.$$

Example 3.18. *To find all of the n-th roots of 1, we notice that*

$$1 = 1e^{i\theta} \quad \text{with } \theta = 0.$$

Therefore,

$$1^{1/n} = \left\{e^{i(2k\pi/n)} : k = 0, 1, ..., n-1\right\}.$$

Definition 3.19. *Given a set $S \subset \mathbb{C}$, a point $z_0 \in \mathbb{C}$ is called an interior point of S if there is $r_0 > 0$ such that*

$$B_{r_0}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\} \subset S.$$

A point $z_0 \in \mathbb{C}$ is called an exterior point of S if there is $r_1 > 0$ such that

$$B_{r_1}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r_1\} \subset \mathbb{C} \backslash S.$$

A point z_0 is a boundary point of S if it is neither an interior point nor an exterior point of S. A point z_0 is an accumulation point or a limit point if for any $r > 0$,

$$B_r(z_0) \cap S \neq \phi.$$

Definition 3.20. *For a set $S \subset \mathbb{C}$, the interior of S consists of all its interior points. We said that S is open if every point in S is an interior point.*

Definition 3.21. *A set S is closed if the complement $\mathbb{C} \backslash S$ is open. The closure of S is the closed set consists of all points of S and its boundary.*

Remark 3.22.

(i) *ϕ and $\mathbb{C}$ are both open and closed.*

(ii) *A set can be neither open nor closed. For example, the set $S = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$.*

(iii) *For the set S in (ii), the interior of S is $\{z \in \mathbb{C} : 1 < |z| < 2\}$, and the closure of S is $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$.*

Definition 3.23. *A set S is connected if it cannot be partitioned into two part $S = S_1 \cup S_2$ for nonempty S_1, S_2 such that*

$$S_1 \subset U \quad \text{and} \quad S_2 \subset V$$

where U and V are disjoint open sets.

4 Functions on Subsets of the Complex Plane

Definition 4.1. *Let S_1 and S_2 be subsets of $\mathbb{C}$. A function f is defined on S_1 if for each $z \in S_1$, there is a unique complex number $f(z) \in S_2$. We write it as*

$$f : S_1 \longrightarrow S_2.$$

The set S_1 is called the domain of f.

Remark 4.2. *A complex function f on S can be represented as*

$$f = f_1 + f_2 i,$$

where f_1 and f_2 are two real-valued functions defined on S.

Here are some examples of functions.

Example 4.3. *$f(z) = z^2$ defined on $\mathbb{C}$. If $z = x + yi$, then*

$$f(z) = \left(x^2 - y^2\right) + 2xyi.$$

Example 4.4. *$f(z) = |z|^2$ defined on $\mathbb{C}$. We have, for $z = x + yi$,*

$$f(z) = x^2 + y^2.$$

Example 4.5. *For $n \in N$, given $n+1$ complex numbers $a_0, a_1, ..., a_n$, then the function*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

is called a polynomial of degree n. P can be defined on the whole $\mathbb{C}$.

Example 4.6. *Let $P(z)$ and $Q(z)$ be two polynomials. The quotient $P(z)/Q(z)$ is called a rational function and is defined at each point z with $Q(z) \neq 0$. For example, the function*

$$R(z) = \frac{z^2+3}{z^3+z^2+5z+5} = \frac{z^2+3}{(z+1)(z^2+5)}$$

is defined on $\mathbb{C}\setminus\left\{-1, \sqrt{5}i, -\sqrt{5}i\right\}$.

Example 4.7. *We know that 0 is the only square root for 0. But for a complex number $z \neq 0$, the square roots of a complex number z are*

$$z^{1/2} = \left\{\sqrt{|z|}e^{i\operatorname{Arg} z/2}, -\sqrt{|z|}e^{i\operatorname{Arg} z/2}\right\},$$

which consists of two values. So $z^{1/2}$ is not a function. But if we particularly choose one of them, say, we define

$$f(z) = \begin{cases} |z|^{1/2}e^{i\operatorname{Arg} z/2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Then f is a function on $\mathbb{C}$. More generally, given any $\theta_0 \in \mathbb{R}$, we can define a function

$$g(z) = \begin{cases} |z|^{1/2}e^{i\theta/2} & \text{if } z = |z|e^{i\theta} \neq 0, \quad \theta \in (\theta_0, \theta_0 + 2\pi], \\ 0 & \text{if } z = 0, \end{cases}$$

which also corresponds to a square root of z.

Example 4.8.

(i) For $z_0 \in \mathbb{C}$, $f_1(z) := z + z_0$, *which is a translation function.*

(ii) For $\theta_0 \in \mathbb{R}$, $f_2(z) := e^{i\theta_0} z$, *which is a rotation function.*

(iii) For $r_0 \in \mathbb{R}$, $f_3(z) := r_0 z$, *which is a scaling function.*

(iv) $f_4(z) := \overline{z}$, *which corresponds to the reflection with respect to the real axis.*

All of these functions are defined on $\mathbb{C}$.

Example 4.9. *Given* $c \in \mathbb{C}$, *the function* e^{cz} *is defined on* $\mathbb{C}$.

Example 4.10. *We define the sine and the cosine for complex numbers by*

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

Also, the hyperbolic sine and the hyperbolic cosine are defined by

$$\cosh z := \frac{e^{z} + e^{-z}}{2} \quad \textit{and} \quad \sinh z := \frac{e^{z} - e^{-z}}{2}.$$

All of these functions are defined on $\mathbb{C}$.

Example 4.11. *The motivation of the definition of the logarithm is to find the inverse of the exponential function. That is, we want to solve the equation*

$$e^z = w$$

for given $w \in \mathbb{C}\backslash\{0\}$. *Suppose that* $w = \rho e^{i\theta}$, $\rho = |w|$, $\theta = \operatorname{Arg} w$, *and* $z = x + yi$, *then the above equation becomes*

$$e^{x+yi} = \rho e^{i\theta}.$$

We have,

$$e^x = \rho \quad \textit{and} \quad e^{iy} = e^{i\theta},$$

which gives

$$x = \ln \rho \quad \textit{and} \quad y = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$

Here $\ln$ *denotes the logarithm for the real numbers. There is a multi-value problem. If we fix* $\alpha_0 \in \mathbb{R}$, *then for each* $\theta \in (-\pi, \pi]$, *we can determine a unique* $k \in \mathbb{Z}$ *such that*

$$\theta + 2k\pi \in (\alpha_0, \alpha_0 + 2\pi].$$

Then we can define

$$\log z := \ln \rho + i\,(\theta + 2k\pi) \quad \textit{such that} \quad \theta + 2k\pi \in (\alpha_0, \alpha_0 + 2\pi],$$

which is a function on $\mathbb{C}\backslash\{0\}$.

Definition 4.12. *(Principal branch of the logarithm) A branch of the logarithm is a continuous function f defined on an open subset U of $\mathbb{C}\backslash\{0\}$ such that*

$$e^{f(z)} = z$$

for all $z \in U$. The principal branch of the logarithm is defined by

$$\log z := \ln|z| + i\operatorname{Arg} z$$

on $\{z : \mathbb{C} : |z| > 0, -\pi < \operatorname{Arg} z < \pi\}$.

Example 4.13. *Given a branch of the logarithm defined on U and a complex number c, we can define the power function z^c to be*

$$z^c = e^{c \log z}.$$

If the principal branch of the logarithm is used, the above definition is called the principal branch of the power function z^c.

Remark 4.14. *In general, given $c \in \mathbb{C}$, z^c might not be defined at 0.*

Example 4.15. *By using the principal branch of the power function z^i,*

$$i^i = e^{i \log i} = e^{i\left(\ln 1 + \frac{\pi i}{2}\right)} = e^{-\pi/2}.$$

5 Continuity of Functions

Definition 5.1. *Given a function f defined on an open set Ω, and z_0 an accumulation point of $\Omega\backslash\{z_0\}$, we call that f has a limit w_0 at z_0 if for all $\varepsilon > 0$, there is $\delta > 0$ such that*

$$|f(z) - w_0| < \varepsilon \quad \textit{for all} \quad z \in \Omega, \quad 0 < |z - z_0| < \delta.$$

And we write it as

$$\lim_{z \to z_0} f(z) = w_0.$$

Proposition 5.2.

(i) If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} f(z) = w_2$, then $w_1 = w_2$.

(ii) If $f(z) = u(z)+iv(z)$, where u and v are real-valued functions, then $\lim_{z\to z_0} f(z) = u_0+v_0 i$ if and only if

$$\lim_{z\to z_0} u(z) = u_0 \quad \textit{and} \quad \lim_{z\to z_0} v(z) = v_0.$$

(iii) If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} g(z) = w_2$, then

$$\lim_{z\to z_0} (f(z) + g(z)) = w_1 + w_2 \quad \textit{and} \quad \lim_{z\to z_0} (f(z)g(z)) = w_1 w_2.$$

If, in addition, $w_2 \neq 0$, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}.$$

Example 5.3. *To show that if $f(z) = i\overline{z}/2$, then*

$$\lim_{z\to 1} f(z) = \frac{i}{2}.$$

Notice that

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\overline{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

We have

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \textit{provided} \quad |z-1| < 2\varepsilon.$$

Example 5.4. *For a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ with $a_0, ..., a_n \in \mathbb{C}$, $n \in \mathbb{N}$, we have the limit*

$$\lim_{z\to z_0} P(z) = a_0 + a_1 z_0 + a_2 z_0^2 + ... + a_n z_0^n.$$

Example 5.5. *Check that if the function*

$$f(z) := z/\overline{z}, \quad z \neq 0,$$

has a limit at 0. Notice that, for $z = x + yi \neq 0$,

$$\operatorname{Re} f = \frac{x^2 - y^2}{x^2 + y^2}.$$

If we approach the origin along the real axis, for all $x \in \mathbb{R}$, we have

$$\operatorname{Re} f(x) = 1.$$

And if we approach the origin along the imaginary axis, for all $y \in \mathbb{R}$, we have

$$\operatorname{Re} f(yi) = -1.$$

Therefore, f does not have a limit at 0.

Example 5.6. *Let $f(z) = 1/\log z$ be defined by using the principal branch of the logarithm on $\Omega = \{z : \mathbb{C} : |z| > 0, -\pi < \operatorname{Arg} z < \pi\}$. To show that*

$$\lim_{z\to 0} f(z) = 0,$$

we recall that for $z \in \Omega$,

$$f(z) = \frac{1}{\log z} = \frac{1}{\ln|z| + i\operatorname{Arg} z}.$$

Hence,

$$|f(z)| \leq \frac{1}{\sqrt{(\ln|z|)^2 + (\operatorname{Arg} z)^2}} \leq -\frac{1}{\ln|z|}$$

for all $z \in \Omega$ with $|z| < 1$. Therefore,

$$|f(z)| < \varepsilon \quad \textit{provided} \quad |z| < \min\left\{1, e^{-1/\varepsilon}\right\}.$$

Definition 5.7. *Given a function f defined on an open set $\Omega \subset \mathbb{C}$, we call that f is continuous at $z_0 \in \Omega$ if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If f is continuous at every point $z \in \Omega$, we call that f is continuous on Ω.

Proposition 5.8. *If f and g are functions on an open set $\Omega \subset \mathbb{C}$ and continuous at $z_0 \in \Omega$, then $f+g$ and fg are both continuous at z_0. Moreover, if $g(z_0) \neq 0$, then f/g is also continuous at z_0.*

Proposition 5.9. *For $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \mathbb{C}$, where Ω_1 and Ω_2 are open sets in $\mathbb{C}$, suppose that f is continuous at $z_0 \in \Omega_1$ and that g is continuous at $f(z_0)$, then the composition $g \circ f$ is continuous at z_0.*

Example 5.10. $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$ *and* $\overline{z}$ *are all continuous functions. If $f : \Omega \to \mathbb{C}$ is continuous on an open set $\Omega \subset \mathbb{C}$, then $|f(z)|$ is also continuous on Ω.*

Example 5.11. *Define the function*

$$f(z) := \begin{cases} z/\overline{z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

By Example 5.5, f does not have a limit at 0. Therefore, f is not continuous at 0.

Example 5.12. *In this example, we will check for what $c \in \mathbb{C}$, the function*

$$f(z) := \begin{cases} z^c, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

is continuous at 0. Here z^c is defined by using the following definition of the logarithm on $\mathbb{C}\backslash\{0\}$:

$$\log z = \ln |z| + i \operatorname{Arg} z.$$

For $z \neq 0$,

$$z^c = e^{c \log z} = e^{c(\ln |z| + i\theta)}.$$

where $\theta = \operatorname{Arg} z$. Suppose that $c = c_1 + c_2 i$, $c_1, c_2 \in \mathbb{R}$, then the above equality becomes

$$z^c = e^{(c_1 \ln |z| - c_2\theta) + i(c_1\theta + c_2 \ln |z|)} = |z|^{c_1} e^{-c_2\theta} e^{i(c_1\theta + c_2 \ln |z|)}.$$

We divide it into three cases.

(i) For $c_1 = 0$, we have

$$z^c = e^{-c_2\theta} e^{ic_2 \ln |z|}, \quad z \neq 0.$$

Taking the modulus of f,

$$|f(z)| = e^{-c_2\theta}.$$

If f is continuous at 0, then $\lim_{z\to 0} f(z) = 0$ by the definition. Equivalently, it holds $\lim_{z\to 0} |f(z)| = 0$. Now, we first approach the origin along the ray with angle 0. We have

$$|f(z)| = 1 \quad \textit{for all } z \textit{ with } \operatorname{Arg} z = 0.$$

Similarly, we can approach the origin along the ray with angle $\pi/2$ and have

$$|f(z)| = e^{-c_2\pi/2} \quad \textit{for all } z \textit{ with } \operatorname{Arg} z = \frac{\pi}{2}.$$

Therefore, if $c_2 \neq 0$, $|f|$ is not continuous at 0, which leads a contradiction. As for the case $c_1 = c_2 = 0$, we have

$$|f(z)| = 1 \quad \textit{for all } z \neq 0.$$

We conclude that f is not continuous at 0 if $c_1 = 0$.

(ii) For $c_1 < 0$, it holds

$$|f(z)| = |z|^{c_1} e^{-c_2\theta} \quad \textit{for all } z \neq 0.$$

In this case, for any θ fixed, since $c_1 < 0$,

$$\lim_{|z|\to 0} |f(z)| = \infty,$$

which implies that f is not continuous at 0.

(iii) For $c_1 > 0$, it holds

$$|f(z)| = |z|^{c_1} e^{-c_2\theta} \quad \textit{for all } z \neq 0.$$

In this case, for any θ fixed, since $c_1 > 0$,

$$\lim_{|z|\to 0} |f(z)| = 0.$$

As a consequence, f is continuous at 0 if $c_1 > 0$.

In summary, f is continuous at 0 if and only if $\operatorname{Re} c > 0$.

Example 5.13. *Let f be the principal square-root function defined by*

$$f(z) = \begin{cases} |z|^{1/2} e^{i\operatorname{Arg} z/2} & \textit{if } z \neq 0, \\ 0 & \textit{if } z = 0. \end{cases}$$

Then f is discontinuous on $S = \{z \in \mathbb{C} : \operatorname{Re} z < 0, \operatorname{Im} z = 0\}$. To see this, given a point $-R \in S$, $R > 0$, we can draw a circle centered at 0 with radius R. If we approach $-R$ along the circle from above, the limit equals to $\sqrt{R}e^{i\pi/2} = \sqrt{R}i$. On the other hand, if we approach $-R$ along the circle from below, the limit equals to $\sqrt{R}e^{-i\pi/2} = -\sqrt{R}i$. Consequently, f does not have a limit at $-R$, and thus is discontinuous there.

6 Differentiability of Functions and the Cauchy-Riemann Equations

Definition 6.1. *Let f be a function on an open set Ω. f is differentiable or holomorphic at $z_0 \in \Omega$ if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. And the limit, if it exists, is called the derivative of f at z_0 and denoted by $f'(z_0)$. The function f is said to be differentiable (or holomorphic) on Ω if it is differentiable at every point of Ω.

Example 6.2. *Let $f(z) = 1/z$ on $\mathbb{C}\backslash\{0\}$. At each $z_0 \neq 0$, we have*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = -\frac{1}{z_0 z}$$

Therefore,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = -\frac{1}{z_0^2}.$$

That is, f is differentiable at $z_0 \neq 0$, and $f'(z_0) = -\frac{1}{z_0^2}$.

Example 6.3. *Let $f(z) = \overline{z}$ on $\mathbb{C}$. For any $z_0 \in \mathbb{C}$, we have*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\overline{z} - \overline{z_0}}{z - z_0} = \frac{\overline{w}}{w},$$

where $w = z - z_0$. Suppose that the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, then equivalently the limit $\lim_{w \to 0} \frac{\overline{w}}{w}$ exists. A same argument in Example 5.5 implies that $\frac{\overline{w}}{w}$ does not have a limit at 0. Therefore, f is not differentiable at every $z_0 \in \mathbb{C}$.

Example 6.4. *Let $f(z) = c$ for some $c \in \mathbb{C}$, then f is differentiable on $\mathbb{C}$ with*

$$f'(z) = 0.$$

Let $g(z) = z^n$ for some $n \in \mathbb{N}$, then g is differentiable on $\mathbb{C}$ with

$$g'(z) = nz^{n-1}.$$

Moreover, for a polynomial $P(z) = a_0 + a_1 z + ... + a_n z^n$, $a_0, a_1, ..., a_n \in \mathbb{C}$, P is differentiable on $\mathbb{C}$ with

$$P'(z) = a_1 + 2a_2 z + ... + na_n z^{n-1}.$$

Proposition 6.5. *If f and g are differentiable functions on Ω, then* *(i)* $f+g$ *is differentiable on* Ω*, and* $(f+g)' = f' + g'$.

(ii) fg *is differentiable on* Ω*, and* $(fg)' = f'g + fg'$.

(iii) If $g(z_0) \neq 0$ *for* $z_0 \in \Omega$*, then* f/g *is differentiable at* z_0*, and*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Moreover, if $f : \Omega_1 \to \Omega_2$ *and* $g : \Omega_2 \to \mathbb{C}$ *are differentiable, then the composition* $g \circ f$ *is differentiable on* Ω_1*, and the chain rule holds*

$$(g(f(z)))' = g'(f(z))\, f'(z).$$

Example 6.6. *Let* $f(z) = |z|^2$ *on* $\mathbb{C}$*. At each* $z_0 \in \mathbb{C}$*, we have*

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{|z|^2 - |z_0|^2}{z - z_0}.$$

By letting $w = z - z_0$*,*

$$|z|^2 = |w + z_0|^2 = (w + z_0)(\overline{w} + \overline{z_0}) = w\overline{w} + w\overline{z_0} + z_0\overline{w} + |z_0|^2.$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{w\overline{w} + w\overline{z_0} + z_0\overline{w}}{w} = \overline{w} + \overline{z_0} + z_0\,\frac{\overline{w}}{w}. \tag{6.1}$$

If $z_0 = 0$*, (6.1) becomes*

$$\frac{f(z) - f(0)}{z - 0} = \overline{w},$$

which implies

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{w \to 0} \overline{w} = 0.$$

Hence, f *is differentiable at* 0 *with* $f'(0) = 0$*. But if* $z_0 \neq 0$*, the last term on the right-hand side of (6.1), i.e.,* $z_0\,\frac{\overline{w}}{w}$*, has no limit at* z_0 *as* $w \to 0$*. Therefore,* f *is not differentiable at every* $z_0 \neq 0$*.*

Remark 6.7. *Example 6.6 illustrates the following facts.*

(i) A function can be differentiable at a point z*, but nowhere else in any neighborhood of that point.*

(ii) By writing a function f *in the form* $f(z) = u(x,y) + iv(x,y)$*,* $z = x + yi$*, we may have* u *and* v *are both differentiable of all orders in variables* (x,y) *at a point* (x_0, y_0)*, but* f *is not differentiable at* $z_0 = x_0 + y_0 i$*.*

(iii) The continuity of a function at a point does not imply the differentiability of the function there.

Proposition 6.8. *If f is differentiable at z_0, then f is continuous at z_0.*

Proof.

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0.$$

$\square$

Theorem 6.9. *Let $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, be defined on a neighborhood of $z_0 = x_0 + y_0 i$. If f is differentiable at z_0, then the partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations*

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

at (x_0, y_0). Moreover, $f'(z_0)$ can be written as

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof. Since $f'(z_0)$ exists, using the definition of $f'(z_0)$ and approaching $z_0 = x_0 + y_0 i$ by $(x_0 + h) + y_0 i$ with $h \in \mathbb{R}$,

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f((x_0 + h) + y_0 i) - f(x_0 + y_0 i)}{h} \\ &= \lim_{h \to 0} \left[\frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \, \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \right] \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned}$$

On the other hand, we can also approach $z_0 = x_0 + y_0 i$ by $x_0 + (y_0 + h)i$ with $h \in \mathbb{R}$, which gives

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(x_0 + (y_0 + h)i) - f(x_0 + y_0 i)}{ih} \\ &= \lim_{h \to 0} \left[-i \, \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h} + \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h} \right] \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned}$$

Then we compete the proof by matching the real and imaginary parts of these two equalities. $\square$

Example 6.10. *Recall that in Example 6.6, $f(z) = |z|^2$ is differentiable only at $z = 0$ with $f'(0) = 0$. Notice that $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, with*

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

It holds that u and v satisfy the Cauchy-Riemann equations at $(0, 0)$. And we have

$$f'(0) = 0 = u_x(0, 0) + iv_x(0, 0).$$

But f cannot be differentiable at any $z \neq 0$ since u and v do not satisfy the Cauchy-Riemann equations there.

Example 6.11. *Let $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, be defined by*

$$f(z) = \begin{cases} \overline{z}^2/z, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

then

$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{-3x^2y + y^3}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Also, $u(0, 0) = v(0, 0) = 0$. Notice that

$$u_x(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and

$$v_y(0, 0) = \lim_{h \to 0} \frac{v(0, h) - v(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

We have $u_x = v_y$ at $(0, 0)$. Similarly, we have $u_y = -v_x = 0$ at $(0, 0)$. That is, the Cauchy-Riemann equations are satisfied at $z = 0$. In contrast, for $z \neq 0$,

$$\frac{f(z) - f(0)}{z - 0} = \left(\frac{\overline{z}}{z}\right)^2$$

does not have a limit as $z \to 0$. To see this, if we approach 0 by $z = \rho e^{i\theta_0}$ for some fixed $\theta_0 \in \mathbb{R}$ and let $\rho \to 0$, we have

$$\left(\frac{\overline{z}}{z}\right)^2 = e^{-4i\theta_0}.$$

We will get different limits as $\rho \to 0$ with different θ_0's.

Theorem 6.12. *Let $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, be defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations at $z_0 = x_0 + y_0 i \in \Omega$, then f is differentiable at z_0 with*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof. By the continuous differentiability of u and v,

$$\begin{aligned} u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) &= u_x(x_0, y_0)h_1 + u_y(x_0, y_0)h_2 + \varphi_1(h)|h|, \\ v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0) &= v_x(x_0, y_0)h_1 + v_y(x_0, y_0)h_2 + \phi_2(h)|h|, \end{aligned}$$

where $\varphi_1(h), \varphi_2(h) \to 0$ as $h \to 0$, $h = h_1 + h_2 i$. Then we have

$$\begin{aligned} &f(z_0 + h) - f(z_0) \\ &= (u_x(x_0, y_0) + iv_x(x_0, y_0))\, h_1 + (u_y(x_0, y_0) + iv_y(x_0, y_0))\, h_2 + (\varphi_1(h) + i\varphi_2(h))\, |h|. \end{aligned}$$

Using the Cauchy-Riemann equations, the above equality becomes

$$\begin{aligned} &f(z_0 + h) - f(z_0) \\ &= (u_x(x_0, y_0) + iv_x(x_0, y_0))\, h_1 + (-v_x(x_0, y_0) + iu_x(x_0, y_0))\, h_2 + (\varphi_1(h) + i\varphi_2(h))\, |h| \\ &= (u_x(x_0, y_0) + iv_x(x_0, y_0))\, (h_1 + h_2 i) + (\varphi_1(h) + i\varphi_2(h))\, |h|. \end{aligned}$$

By passing to the limit $h \to 0$, we complete the proof. □

Example 6.13. *Recall that in Example 6.11, $f(z) = u(x,y) + iv(x,y)$, $z = x + yi$, defined by*

$$f(z) = \begin{cases} \overline{z}^2/z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Though u and v satisfy the Cauchy-Riemann equations at $(x,y) = (0,0)$, the partial derivatives of u and v are not continuous at $(0,0)$. The assumptions of Theorem 6.12 do not holds.

Example 6.14. *Consider the function $f(z) = e^z = e^x(\cos y + i \sin y)$, where $z = x + yi$. Then we have $f(z) = u(x,y) + iv(x,y)$ with*

$$u(x,y) = e^x \cos y \quad \text{and} \quad v(x,y) = e^x \sin y.$$

Notice that u and v are both continuously differentiable and satisfy

$$u_x = e^x \cos y = v_y \quad \text{and} \quad u_y = -e^x \sin y = -v_x.$$

for all $(x,y) \in \mathbb{R}^2$. Therefore, f is differentiable on $\mathbb{C}$ with

$$f' = u_x + iv_x = e^x \cos y + ie^x \sin y.$$

Note that $f'(z) = f(z)$ for all $z \in \mathbb{C}$.

Example 6.15. *Let $f(z) = x^3 + i(1-y)^3$, $z = x + yi$. Then $f(z) = u(x,y) + iv(x,y)$ with*

$$u(x,y) = x^3 \quad \text{and} \quad v(x,y) = (1-y)^3.$$

First, notice that u and v are continuously differentiable on $\mathbb{R}^2$. As for the Cauchy-Riemann equations,

$$\begin{aligned} u_x &= 3x^2, & u_y &= 0, \\ v_x &= 0, & v_y &= -3(1-y)^2. \end{aligned}$$

Then we always have $u_y = -v_x$. But $u_x = v_y$ only if $(x,y) = (0,1)$. Therefore, f is differentiable only at $z = i$ with

$$f'(i) = u_x(0,1) + iv_x(0,1) = 0.$$

Example 6.16. *Let $f(z) = \sin x \cosh y + i \cos x \sinh y$, $z = x + yi \in \mathbb{C}$. Then $f = u + iv$ with*

$$u(x,y) = \sin x \cosh y \quad \text{and} \quad v(x,y) = \cos x \sinh y.$$

Since u and v are continuously differentiable and satisfy

$$u_x = \cos x \cosh y = v_y \quad \text{and} \quad u_y = \sin x \sinh y = -v_x$$

everywhere, we conclude that f is differentiable on $\mathbb{C}$ with

$$f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y.$$

Theorem 6.17. *If $f'(z) = 0$ on an open connected set Ω, then f is a constant on Ω.*

Lemma 6.18. *If an open set Ω is connected, then it is polygonally connected. That is, for any $z_1, z_2 \in \Omega$, z_1 and z_2 can be connected by a polygonal line consisting of finitely many line segments in Ω.*

Proof. If $\Omega = \phi$, then there is nothing to prove. By choosing a point $z_0 \in \Omega$, we define the set

$$S = \{z \in \Omega : z \text{ can be connected to } z_0 \text{ by a polygonal line}\}.$$

Given a point $z_1 \in S$, since Ω is open, there is $\varepsilon_1 > 0$ small enough such that $B_{\varepsilon_1}(z_1) \subset \Omega$. Notice that any point in $B_{\varepsilon_1}(z_1)$ can be connected to z_1 by a line segment. Thus, $B_{\varepsilon_1}(z_1) \subset S$, which implies that S is open.

Suppose that $\Omega \backslash S \neq \phi$, say, there is $z_2 \in \Omega \backslash S$. Again, we have $B_{\varepsilon_2}(z_2) \subset S$ for some $\varepsilon_2 > 0$. All point in $B_{\varepsilon_2}(z_2)$ do not belong to S. Otherwise, z_2 can be polygonally connected to z_0. Thus, $\Omega \backslash S$ is also open, which leads a contradiction. We conclude that $\Omega \backslash S = \phi$, i.e., $S = \Omega$. Therefore, for any two points $w_1, w_2 \in \Omega$, they can be connected by a polygonal line in Ω by combining one polygonal line connecting z_0 to w_1 and another one connecting z_0 to w_2. $\square$

Proof of Theorem 6.17. Let $f(z) = u(x, y) + iv(x, y)$ for $z = x + yi$. Since $f'(z) = 0$,

$$f'(z) = u_x(x, y) + iv_x(x, y) = 0.$$

In view of the Cauchy-Riemann equations, we have

$$u_x = u_y = v_x = v_y = 0 \quad \text{on } \Omega.$$

Next, if $z_1, z_2 \in \Omega$ such that the line segment L between z_1 and z_2 lie in Ω, we will show that $f(z)$ is a constant on L. L can be parametrized by

$$L = \left\{z_1 + sw : s \in \big[0, |z_2 - z_1|\big]\right\},$$

where $w = w_1 + w_2 i = \dfrac{z_2 - z_1}{|z_2 - z_1|}$ is the unit vector in the direction from z_1 to z_2. Now, we consider the restriction of u on L, i.e., $u(x_1 + w_1 s, y_1 + w_2 s)$, where $z_1 = x_1 + y_1 i$. We have

$$\frac{d}{ds} u(x_1 + w_1 s, y_1 + w_2 s) = \nabla u\Big|_{(x_1 + w_1 s, y_1 + w_2 s)} \cdot w,$$

where $\nabla u = (u_x, u_y)$ is the gradient of u. Since $u_x = u_y = 0$, it follows that

$$\frac{d}{ds} u(x_1 + w_1 s, y_1 + w_2 s) = 0 \quad \text{on } \big[0, |z_2 - z_1|\big].$$

This gives u is a constant on L. Since there is always a finite number of line segments connecting any two points in Ω, u is a constant on Ω. Similarly, by applying the same arguments to v, v is a constant on Ω. Therefore, f is a constant on Ω. $\square$

Example 6.19. *Suppose that f and $\overline{f}$ are both differentiable on an open connected set Ω, we show that f must be a constant.*

By writing $f(z) = u(x, y) + iv(x, y)$, $z = x + yi$, we have $\overline{f(z)} = u(x, y) - iv(x, y)$. Since f is differentiable on Ω, the Cauchy-Riemann equations

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x \quad \textit{hold on } \Omega.$$

Since $\overline{f}$ is also differentiable on Ω, the Cauchy-Riemann equations

$$u_x = -v_y \quad \textit{and} \quad u_y = v_x \quad \textit{hold on } \Omega.$$

Therefore, we have $u_x = u_y = v_x = v_y = 0$ on Ω, which implies $f'(z) = 0$ on Ω. By Theorem 6.17, f is a constant.

Example 6.20. *Suppose that f is differentiable on an open connected set Ω. If $|f|$ is a constant on Ω, we show that f must be a constant.*

If $|f| = 0$ on Ω, then it follows that $f = 0$ on Ω. Now, we assume that $|f| = c \neq 0$ on Ω, we have

$$f(z)\overline{f(z)} = |f|^2 = c^2 \neq 0.$$

Notice that $f \neq 0$ on Ω. And hence

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

is differentiable on Ω. The last example implies that f is a constant.

7 Analyticity and Harmonicity

Definition 7.1. *Let f be a function defined on an open set $\Omega \subset \mathbb{C}$. f is called analytic at a point $z_0 \in \Omega$ if f is differentiable on a neighborhood $B_\varepsilon(z_0) \subset \Omega$ for some $\varepsilon > 0$. If f is differentiable on Ω, then we also called f is analytic on Ω. Moreover, if f is differentiable or analytic on Ω, we call f an entire function.*

Remark 7.2. *In some literatures, the analyticity f is defined as follows: f is called analytic at a point $z_0 \in \Omega$ if there is a power series $\sum a_n(z - z_0)^n$ with a radius of convergence such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in B_\varepsilon(z_0) \subset \Omega$ for some $\varepsilon > 0$. And f is called analytic on Ω if f has a power series expansion at every point in Ω. It can be proved that the two definitions of analyticity are equivalent.

Example 7.3. *$f(z) = 1/z$ is differentiable on $\mathbb{C}\backslash\{0\}$with $f'(z) = -1/z^2$ for $z \neq 0$. So f is analytic on $\mathbb{C}\backslash\{0\}$. $g(z) = |z|^2$ is differentiable only at $z = 0$. Thus, g is not analytic anywhere. Finally, we have that every polynomial is an entire function.*

Theorem 7.4. *Suppost that $f = u + iv$ is analytic on an open set Ω. Then u and v are harmonic functions on Ω.*

Proof. To show this, we need to use the fact that if a complex function is analytic at a point, then its real and imaginary parts have continuous partial derivatives of all orders there.

Since u and v satisfy the Cauchy-Riemann equations, it holds that

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{on } \Omega.$$

Therefore,

$$u_{xx} = v_{xy} \quad \text{and} \quad u_{yy} = -v_{xy} \quad \text{on } \Omega.$$

We get

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0 \quad \text{on } \Omega.$$

That is, u is a harmonic function on Ω. The arguments for v is similar. $\square$

8 Integrals

Definition 8.1. *Let w be a complex-valued function of a real variable t, written as*

$$w(t) = u(t) + iv(t),$$

for some real-valued functions u and v. The derivative of w is defined by

$$\frac{d}{dt}w(t) = w'(t) = u'(t) + iv'(t),$$

provided that u and v are differentiable. And the definite integral of w over an interval $[a, b]$ is defined by

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the integrals on the right-hand side exist.

Example 8.2.

$$\begin{aligned}\int_0^{\pi/4} e^{it}dt &= \int_0^{\pi/4} (\cos t + i\sin t)\, dt = \int_0^{\pi/4} \cos t dt + i\int_0^{\pi/4} \sin t dt \\ &= \sin t\Big|_0^{\pi/4} + i(-\cos t)\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2} + \left(1 - \frac{\sqrt{2}}{2}\right) i.\end{aligned}$$

Proposition 8.3. *If $w(t) = u(t)+iv(t)$ is a complex-valued function on $[a, b]$, and $W'(t) = w(t)$, i.e., $W(t) = U(t) + iV(t)$ with $U'(t) = u(t)$, $V'(t) = v(t)$, then*

$$\int_a^b w(t)dt = W(b) - W(a)$$

Proof. By the fundamental theorem of calculus,

$$\int_a^b u(t)dt = U(b) - U(a) \quad \text{and} \quad \int_a^b v(t)dt = V(b) - V(a).$$

$\square$

Example 8.4. *Since*

$$\frac{d}{dt}\frac{e^{it}}{i} = e^{it},$$

we have

$$\int_0^{\pi/4} e^{it}dt = \frac{e^{it}}{i}\bigg|_0^{\pi/4} = -ie^{it}\bigg|_0^{\pi/4} = -i\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - 1\right) = \frac{\sqrt{2}}{2} + \left(1 - \frac{\sqrt{2}}{2}\right)i.$$

Definition 8.5. *A (parametrized) curve γ is a set*

$$\gamma = \{z = z(t) = x(t) + y(t)i : t \in [a, b]\}, \tag{8.1}$$

where $x(t)$ and $y(t)$ are continuous real functions on $[a, b]$. γ is called a simple curve or a Jordan curve if it does not intersect itself, that is, $z(t_1) \neq z(t_2)$ unless $t_1 = t_2$. γ is called a simple closed curve if it does not intersect itself except for $z(a) = z(b)$. If $x(t)$ and $y(t)$ are continuously differentiable on $[a, b]$, then γ is called a smooth curve. γ is called a piecewise smooth curve if there are points

$$a = a_0 < a_1 < ... < a_n = b,$$

such that $x(t)$ and $y(t)$ are smooth in each interval $[a_{k-1}, a_k]$, $k = 1, ..., n$.

Remark 8.6.

(i) The set defined by (8.1) is only a geometric object, which does not have a direction. But if we parametrize it by the parametrization $z(t) = x(t)+y(t)i$, then it is assigned a direction.

(ii) The length of a smooth curve $\gamma = \{z = z(t) : t \in [a, b]\}$ is

$$\text{length}(\gamma) = \int_a^b |z'(t)|dt.$$

If γ is only piecewise smooth, its length is the sum of the lengths of its smooth parts.

Definition 8.7. *Given curve γ defined in (8.1), we use $-\gamma$ to denote the same set of points of (8.1) but with reverse direction, say,*

$$-\gamma = \{z(a + b - t) : t \in [a, b]\}.$$

Definition 8.8. *Two parametrization $z_1(t) : [a, b] \to \mathbb{C}$ and $z_2(t) : [c, d] \to \mathbb{C}$ are called equivalent if there exists a continuously differentiable bijection $s \mapsto t(s)$ from $[c, d]$ to $[a, b]$ such that $t'(s) > 0$ and*

$$z_2(s) = z_1(t(s)).$$

Example 8.9. *Here are some examples of curves.*

(i) The polygonal line defined by

$$z(t) = \begin{cases} t + it, & t \in [0, 1], \\ t + i, & t \in [1, 2], \end{cases}$$

is a piecewise smooth curve.

(ii) The unit circle with parametrization

$$z(\theta) = e^{i\theta}, \quad \theta \in [0, 2\pi],$$

is a simple closed smooth curve.

(iii) If γ be the unit circle defined in (ii). Then $-\gamma$ can be defined by the parametrization

$$z(\theta) = e^{-i\theta}, \quad \theta \in [0, 2\pi],$$

(iv) Given $m \in \mathbb{Z}\backslash\{0\}$, the curve defined by

$$z(\theta) = e^{im\theta}, \quad \theta \in [0, 2\pi],$$

winds around the origin m times counterclockwise if $m > 0$. If $m < 0$, it winds around the origin m times clockwise.

Formal Calculation. *Suppose that there is a differentiable function $F = U + iV$ such that $F' = f$ on Ω, we have, by using the Cauchy-Riemann equations,*

$$\begin{aligned}
&\frac{d}{dt}F(z(t)) = \frac{d}{dt}F(x(t) + iy(t)) \\
&= \frac{d}{dt}U(x(t), y(t)) + i\frac{d}{dt}V(x(t), y(t)) \\
&= U_x(x(t), y(t))x'(t) + U_y(x(t), y(t))y'(t) + iV_x(x(t), y(t))x'(t) + iV_y(x(t), y(t))y'(t) \\
&= U_x(x(t), y(t))x'(t) - V_x(x(t), y(t))y'(t) - iV_x(x(t), y(t))x'(t) + iU_x(x(t), y(t))y'(t) \\
&= (U_x(x(t), y(t)) + iV_x(x(t), y(t)))(x'(t) + iy'(t)) \\
&= f(z(t))z'(t). \qquad (8.2)
\end{aligned}$$

Then Proposition 8.3 implies that

$$\int_a^b f(z(t))z'(t)dt = F(z(b)) - F(z(a)).$$

Definition 8.10. *Let γ be a smooth curve with parametrization $z(t)$, $t \in [a, b]$. If f is a continuous function on an open set Ω containing γ, we define*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

If γ is only piecewise smooth, which is smooth on intervals $[a_{k-1}, a_k]$, $k = 1, ..., n$, where $a = a_0 < a_1 < ... < a_n = b$, then

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} f(z(t))z'(t)dt.$$

Remark 8.11. *The definition of integrals of functions along a curve γ is independent of the choice of the parametrization for γ. For*

$$\gamma = \{z = z_1(t) : t \in [a, b]\}$$

and an equivalent parametrization $z_2 : [c,d] \to \mathbb{C}$ with

$$z_2(s) = z_1(t(s)), \quad t'(s) > 0,$$

we have

$$\int_a^b f(z_1(t))z_1'(t)dt = \int_c^d f(z_1(t(s)))z_1'(t(s))t'(s)ds = \int_c^d f(z_2(s))z_2'(s)ds.$$

Proposition 8.12.

(i) If $c_1, c_2 \in \mathbb{C}$, then

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z))\, dz = c_1 \int_\gamma f_1(z)dz + c_2 \int_\gamma f_2(z)dz.$$

(ii)

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$$

(iii)

$$\left|\int_\gamma f(z)dz\right| \le \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

Proof. Without loss of generality, we assume that γ is smooth. Part (i) follows the linearity of the Riemann integrals. For (ii), if

$$\gamma = \{z = z(t) : t \in [a,b]\},$$

we have

$$\begin{aligned}\int_{-\gamma} f(z)dz &= \int_a^b f(z(a+b-s))(z(a+b-s))'ds = -\int_a^b f(z(a+b-s))z'(a+b-s)ds \\ &= \int_b^a f(z(t))z'(t)dt = -\int_a^b f(z(t))z'(t)dt = -\int_\gamma f(z)dz.\end{aligned}$$

For (iii),

$$\left|\int_\gamma f(z)dz\right| \le \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|dt = \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma).$$

□

Example 8.13. *To evaluate*

$$\int_\gamma \frac{dz}{z},$$

where $\gamma = \left\{z = e^{i\theta} : \theta \in [0,\pi]\right\}$, it holds

$$\int_\gamma \frac{dz}{z} = \int_0^\pi \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = i\int_0^\pi d\theta = \pi i.$$

Example 8.14. *Let γ be a smooth curve with parametrization $z(t)$, $t \in [a, b]$. Notice that*

$$\frac{d}{dt}(z(t))^2 = 2z(t)z'(t),$$

it holds

$$\int_\gamma z dz = \int_a^b z(t)z'(t)dt = \frac{1}{2}(z(t))^2\Big|_a^b = \frac{1}{2}\left((z(b))^2 - (z(a))^2\right).$$

Example 8.15. *Let γ_1 be the polygonal line starting from 0 to i, and then coming from i to $1+i$, then*

$$\int_{\gamma_1} \left(y - x - 3x^2 i\right) dz = \int_0^1 ti dt + \int_0^1 \left(1 - t - 3t^2 i\right) dt = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2} - \frac{i}{2}.$$

Let γ_2 be the line segment from 0 to $1+i$, then

$$\int_{\gamma_2} \left(y - x - 3x^2 i\right) dz = \int_0^1 \left(t - t - 3t^2 i\right)(1+i)dt = 1 - i.$$

Example 8.16. *Let γ be the semicircular path parametrized by*

$$z(\theta) = 3e^{i\theta}, \quad \theta \in [0, \pi],$$

and $f(z) = z^{1/2}$ be defined by using the branch of the logarithm

$$\log z = \ln|z| + i\theta, \quad \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Then

$$\int_\gamma f(z)dz = \int_0^\pi \left(3e^{i\theta}\right)^{1/2} 3ie^{i\theta}d\theta = \int_0^\pi e^{\frac{1}{2}(\ln 3 + i\theta)}3ie^{i\theta}d\theta = 3\sqrt{3}i\int_0^\pi e^{3i\theta/2}d\theta$$
$$= 2\sqrt{3}\left(e^{3\pi i/2} - 1\right) = -2\sqrt{3}\,(1+i)\,.$$

Example 8.17. *Let γ be the unit circle with parametrization*

$$z(\theta) = e^{i\theta}, \quad \theta \in [-\pi, \pi].$$

And let $f(z) = z^{-1+i}$ be defined by using the principal branch of the logarithm. Notice that f is defined only for $\theta \in (-\pi, \pi)$ on γ. On the other hand, for $\theta \in (-\pi, \pi)$,

$$f(z(\theta))z'(\theta) = e^{(-1+i)(i\theta)}ie^{i\theta} = ie^{-\theta}$$

is continuous in $(-\pi, \pi)$ and has limits at $\theta = \pm\pi$. Thus, the (improper) integral exists and

$$\int_\gamma f(z)dz = \int_{-\pi}^{\pi} ie^{-\theta}d\theta = i\left(e^\pi - e^{-\pi}\right).$$

Example 8.18. *To estimate* $\left|\int_\gamma \frac{z-2}{z^4+1}dz\right|$, *where* γ *is the arc of the circle* $|z| = 2$ *from* $z = 2$ *to* $z = 2i$, *we have*

$$\left|\frac{z-2}{z^4+1}\right| \leq \frac{|z|+2}{|z|^4-1} = \frac{4}{15} \quad \textit{for } |z| = 2.$$

Therefore,

$$\left|\int_\gamma \frac{z-2}{z^4+1}dz\right| \leq \frac{4}{15}\,\mathrm{length}(\gamma) = \frac{4\pi}{15}.$$

Example 8.19. *Let* γ_R *be the semicircle parametrized by*

$$z(\theta) = Re^{i\theta}, \quad \theta \in [0, \pi].$$

We are going to show that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)}dz = 0$$

without actually evaluating the integral. Notice that

$$\left|\frac{z+1}{(z^2+4)(z^2+9)}\right| \leq \frac{|z|+1}{(|z|^2-4)(|z|^2-9)} = \frac{R+1}{(R^2-4)(R^2-9)} \quad \textit{on } \gamma_R, \quad R > 3.$$

Thus, for $R > 3$,

$$\left|\int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)}dz\right| \leq \frac{R+1}{(R^2-4)(R^2-9)} \cdot \pi R \longrightarrow 0 \quad \textit{as } R \to \infty.$$

As a consequence, we obtain the limit

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{z+1}{(z^2+4)(z^2+9)}dz = 0.$$

9 Antiderivatives and Independence of Path

Definition 9.1. *Let* f *be a function defined on an open connected set* Ω. *If there is a differentiable function* F *such that* $F' = f$ *on* Ω, *then we call* F *an antiderivative of* f.

Remark 9.2. *Antiderivatives of a given function are unique up to a constant.*

Theorem 9.3. *Let* f *be a continuous function on an open connected set* Ω. *If* f *has an antiderivative* F *on* Ω, *then for any piecewise smooth curve* γ *from* z_1 *to* z_2 *for some* $z_1, z_2 \in \Omega$, *we have*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

Remark 9.4. *In particular, if* f *has an antiderivative, then the integral of* f *along any piecewise smooth closed curve equals to* 0.

Proof. Let γ be parametrized by $z(t) : [a, b] \to \Omega$. If γ is smooth, then, as in (8.2),

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

If γ is only piecewise smooth, let z be smooth on each interval $[a_{k-1}, a_k]$, $k = 1, ..., n$, where $a = a_0 < a_1 < ... < a_n = b$. Then

$$\int_\gamma f(z)dz = \sum_{k=1}^n [F(z(a_k)) - F(z(a_{k-1}))] = F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

□

Theorem 9.5. *Let f be a continuous function on an open connected set Ω. If*

$$\int_\gamma f(z)dz = 0$$

for all piecewise smooth closed curve in Ω, then f has an antiderivative.

Lemma 9.6. *Under the same assumption as in Theorem 9.5, given $z_1, z_2 \in \Omega$,*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

for any piecewise smooth curves γ_1 and γ_2 from z_1 to z_2.

Proof. Let γ_1 and γ_2 be two piecewise smooth curves from z_1 to z_2, $z_1, z_2 \in \Omega$, we have

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = \int_{\gamma_1 \cup (-\gamma_2)} f(z)dz = 0.$$

□

Proof of Theorem 9.5. Fix $z_0 \in \Omega$. In view of Lemma 9.6, we can define a function

$$F(z) = \int_{\gamma_{z_0,z}} f(w)dw, \quad z \in \Omega,$$

where $\gamma_{z_0,z}$ is any smooth curve from z_0 to z. Then, for each $z \in \Omega$ and $h \in \mathbb{C}$ with $|h|$ sufficiently small,

$$F(z+h) - F(z) = \int_{\gamma_{z_0,z+h}} f(w)dw - \int_{\gamma_{z_0,z}} f(w)dw = \int_{\gamma_{z,z+h}} f(w)dw,$$

where $\gamma_{z,z'}$ denotes a curve lying in Ω from z to z'. Since the integration is independent of the choice of curves, we have

$$F(z+h) - F(z) = \int_0^1 f(z+ht)hdt,$$

and hence

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_0^1 [f(z+ht)-f(z)]\, hdt = \int_0^1 [f(z+ht)-f(z)]\, dt.$$

Notice that by the continuity of f,

$$\left|\int_0^1 [f(z+ht)-f(z)]\, dt\right| \le \sup_{t\in[0,1]} |f(z+ht)-f(z)| \to 0 \quad \text{as } h\to 0,$$

which implies

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

$\square$

Remark 9.7. *To summarize, the following three statements are equivalent.*

(i) f has an antiderivative.

(ii) Integration of f from one point to another is independent of the choice of curves.

(iii) Integrals of f along closed curves have value 0.

Example 9.8. *The continuous function $f(z) = e^{\pi z}$ has an antiderivative $F(z) = e^{\pi z}/\pi$ on $\mathbb{C}$. Hence, for any piecewise smooth curve γ from i to $i/2$, we have*

$$\int_\gamma e^{\pi z} dz = \left.\frac{e^{\pi z}}{\pi}\right|_i^{i/2} = \frac{1+i}{\pi}.$$

Example 9.9. *The function $f(z) = 1/z^2$ has an antiderivative $F(z) = -1/z$ on $\mathbb{C}\backslash\{0\}$. Hence,*

$$\int_\gamma \frac{dz}{z^2} = 0,$$

where γ is the unit circle parametrized by $z(\theta) = e^{i\theta}$, $\theta \in [-\pi, \pi]$. As for the function $g(z) = 1/z$, the integral of g along γ cannot be evaluated in a similar way. Notice that given a branch of the logarithm, $G(z) = \log z$ is an antiderivative of $1/z$ on the domain where the logarithm is defined. But the domain of G cannot contain the whole curve γ.

Example 9.10. *To evaluate the integral*

$$\int_\gamma \frac{dz}{z},$$

where γ is defined as in Example 9.9, we can divide γ into two parts: γ_1 is the right half from $-i$ to i parametrized by $z_1(\theta) = e^{i\theta}$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and γ_2 is the left half from i to $-i$ parametrized by $z_2(\theta) = e^{i\theta}$, $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. For γ_1, we know that the principal branch of the logarithm is an antiderivative of $1/z$ on an open set containing γ_1. Thus,

$$\int_{\gamma_1} \frac{dz}{z} = \log z\Big|_{-i}^{i} = \log i - \log(-i) = \frac{\pi i}{2} - \left(-\frac{\pi i}{2}\right) = \pi i,$$

where we used the principal branch of the logarithm here. As for γ_2, by using the branch of the logarithm

$$\log z = \ln|z| + i\theta, \quad \textit{where } \theta \in \arg z, \quad \theta \in (0, 2\pi),$$

defined on $\{|z| > 0, \operatorname{Arg} z \neq 0\}$, *we have*

$$\int_{\gamma_1} \frac{dz}{z} = \log z \Big|_{i}^{-i} = \log(-i) - \log i = \frac{3\pi i}{2} - \frac{\pi i}{2} = \pi i.$$

Therefore,

$$\int_{\gamma} \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} = 2\pi i.$$

Example 9.11. *Let f be the square-root function on* $\left\{|z| > 0, \operatorname{Arg} z \neq -\frac{\pi}{2}\right\}$ *defined by*

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} = |z|^{1/2} e^{i\theta/2} \quad \textit{if } z = |z|e^{i\theta}, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

That is, the power function is defined by using the following branch of the logarithm

$$\log z = \ln|z| + i\theta, \quad \textit{where } \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

If γ is a curve from -3 to 3 lying above the real axis except for the endpoints, noticing that

$$\left(z^{3/2}\right)' = \frac{3}{2} z^{1/2},$$

we have

$$\int_{\gamma} f(z)dz = \frac{2}{3} z^{3/2} \Big|_{-3}^{3} = 2\sqrt{3}\,(1+i)\,.$$

10 Integration of Analytic Functions on Closed Loops

Theorem 10.1 (Cauchy-Goursat theorem)**.** *If f is analytic at all points interior to and on a simple closed curve γ, then*

$$\int_{\gamma} f(z)dz = 0.$$

Theorem 10.2. *If f is differentiable on an open set Ω, and γ is the boundary of an rectangle contained in Ω, then*

$$\int_{\gamma} f(z)dz = 0.$$

Proof. Let R_0 be the closed rectangle with boundary γ. Assume that $\gamma_0 = \gamma = l_1 \cup l_2 \cup l_3 \cup l_4$, counterclockwise oriented. Let z_k be the midpoint of l_k, $k = 1, ..., 4$. By connecting z_1 and z_3, and connecting z_2 and z_4, we obtain four smaller rectangles with boundaries $\gamma_{1,1}$, $\gamma_{1,2}$, $\gamma_{1,3}$ and $\gamma_{1,4}$. We assume that $\gamma_{1,j}$, $j = 1, ..., 4$, are all counterclockwise oriented. We have

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_{1,1}} f(z)dz + \int_{\gamma_{1,2}} f(z)dz + \int_{\gamma_{1,3}} f(z)dz + \int_{\gamma_{1,4}} f(z)dz.$$

By the triangle inequality,

$$\left|\int_{\gamma_0} f(z)dz\right| = \left|\int_{\gamma_{1,1}} f(z)dz\right| + \left|\int_{\gamma_{1,2}} f(z)dz\right| + \left|\int_{\gamma_{1,3}} f(z)dz\right| + \left|\int_{\gamma_{1,4}} f(z)dz\right|.$$

There must be a $j \in \{1, 2, 3, 4\}$ such that

$$\left|\int_{\gamma_0} f(z)dz\right| \leq 4\left|\int_{\gamma_{1,j}} f(z)dz\right|.$$

Let $\gamma_1 = \gamma_{1,j}$ with j such that the last inequality holds, and R_1 be the closed rectangle with boundary γ_1. We can repeat the same process. For γ_n given, we divide the rectangle into four parts with boundary $\gamma_{n+1,j}$, $j = 1, ..., 4$. And we can choose a j such that

$$\left|\int_{\gamma_n} f(z)dz\right| \leq 4\left|\int_{\gamma_{n+1,j}} f(z)dz\right|.$$

And then we denote $\gamma_{n+1} = \gamma_{n+1,j}$ with j such that the last inequality holds. We obtain a sequence of rectangles R_n with boundaries γ_n, $n \in \mathbb{N} \cup \{0\}$, such that

$$R_0 \supset R_1 \supset ... \supset R_n \supset ... \tag{10.1}$$

and

$$\left|\int_{\gamma_0} f(z)dz\right| \leq 4^n\left|\int_{\gamma_n} f(z)dz\right|. \tag{10.2}$$

Since R_n's are compact satisfying (10.1) with $\mathrm{diam}(R_n) \to 0$ as $n \to \infty$, there is a unique $z_0 \in \Omega$ such that $z_0 \in R_n$ for all n. Since f is differentiable at z_0,

$$\lim_{z \to z_0}\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| = 0.$$

On each γ_n, since constants and polynomials have antiderivatives,

$$\int_{\gamma_n} f(z)dz = \int_{\gamma_n} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right] dz = \int_{\gamma_n} \left[\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right](z - z_0)dz.$$

Therefore,

$$\left|\int_{\gamma_n} f(z)dz\right| \leq \sup_{z\in\gamma_n}\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \sup_{z\in\gamma_n}|z - z_0| \cdot \mathrm{length}(\gamma_n).$$

Notice that

$$\sup_{z\in\gamma_n} |z - z_0| \leq 2^{-n} L,$$

where L is the length of the diagonal of R_0, and

$$\text{length}(\gamma_n) = 2^{-n}\text{length}(\gamma).$$

Therefore,

$$\left|\int_{\gamma_n} f(z)dz\right| \leq 4^{-n} L \cdot \text{length}(\gamma) \sup_{z\in\gamma_n} \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right|. \tag{10.3}$$

Combining (10.2) and (10.3), we obtain

$$\left|\int_{\gamma} f(z)dz\right| \leq L \cdot \text{length}(\gamma) \sup_{z\in\gamma_n} \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

□

Theorem 10.3. *If f is differentiable on an open disc Ω, then*

$$\int_\gamma f(z)dz = 0$$

for all closed curve γ in Ω.

Proof. Without loss of generality, we may assume that the disc is centered at 0. Define

$$F(z) = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,$$

where γ_1 is the line segment from 0 to $\operatorname{Re} z$, and γ_2 is the line segment from $\operatorname{Re} z$ to z. By Theorem 10.2, for $h = h_1 + h_2 i$ with $|h|$ sufficiently small,

$$F(z+h) - F(z) = \int_\gamma f(z)dz,$$

where γ is the polygonal line starting from z to $z + h_1$ and then from $z + h_1$ to $z + h$. Using a similar argument in the proof of Theorem 9.5,

$$\begin{aligned}
&\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \\
= &\left|\frac{1}{h}\left[\int_0^1 (f(z + h_1 t) - f(z))\, h_1 dt + \int_0^1 (f(z + h_1 + ih_2 s) - f(z))\, ih_2 ds\right]\right| \\
\leq &\sup_{t\in[0,1],\, s\in[0,1]} [|f(z + h_1 t) - f(z)| + |f(z + h_1 + ih_2 s) - f(z)|] \\
\longrightarrow &\ 0 \quad \text{as } h \to 0.
\end{aligned}$$

That is, F is an antiderivative of f, and hence the theorem follows. □

Proof of Theorem 10.1. Let K be the closed region bounded by γ. By the assumption, f is differentiable on some bounded open set Ω containing K, and we have $\mathrm{dist}(K, \partial\Omega) > 3\varepsilon$ for some $\varepsilon > 0$, where $\partial\Omega$ is the boundary of Ω. We may assume $l_0 = \gamma$ is counterclockwise oriented. And let l_1 be a simple closed curve lying in the interior of l_0, also counterclockwise oriented, such that $\mathrm{dist}(z, l_0) < \varepsilon$ for every $z \in l_1$. And then we slice the strip bounded by l_0 and l_1 into small pieces. Each piece is contained in a disc with radius 2ε contained in Ω. By Theorem 10.3, summing the integrals on the boundaries of all the pieces, we obtain

$$\int_{l_0} f(z)dz + \int_{-l_1} f(z)dz = 0.$$

Here the integrals on common edges are cancelled out. The above equality shows that

$$\int_{l_0} f(z)dz = \int_{l_1} f(z)dz.$$

We can continue in this manner to obtain a sequence of simple closed curves l_0, l_2, ..., l_n, ..., with $\mathrm{length}(l_n) \to 0$ as $n \to \infty$, and

$$\int_{l_n} f(z)dz = \int_{l_{n+1}} f(z)dz$$

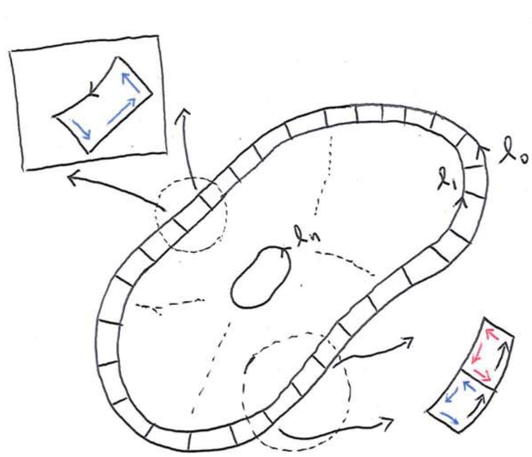
for all $n \in \mathbb{N} \cup \{0\}$. Therefore,

$$\int_{l_0} f(z)dz = \int_{l_n} f(z)dz$$

for all $n \in \mathbb{N}$. In addition, for l_n, we have

$$\left| \int_{l_n} f(z)dz \right| \le \max_K f \cdot \mathrm{length}(l_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, we complete the proof.



$\square$

Definition 10.4. *A connected set Ω is called simply connected if for any two curves γ_0 and γ_1 in Ω, parametrized by $z_0(t)$ and $z_1(t)$, $t \in [a,b]$, respectively, with same endpoints, there exists a class of curves γ_s parametrized by $z_s(t)$, $t \in [a,b]$, for $s \in [0,1]$, such that*

$$z_s(a) = z_0(a) = z_1(a) \quad \text{and} \quad z_s(b) = z_0(b) = z_1(b) \quad \text{for all } s \in [0,1],$$

and $z_s(t)$ is jointly continuous in $(s,t) \in [0,1] \times [a,b]$. If a connected set Ω is not simply connected, then it is called multiply connected.

Remark 10.5. *There is an equivalent definition for simply connected sets as follows. A connected set Ω is called simply connected if every simple closed curve in Ω encloses only points in Ω.*

Theorem 10.6. *If f is analytic on an open simply connected domain Ω, then*

$$\int_\gamma f(z)dz = 0$$

for all closed curve γ lying in Ω.

Proof. It suffices to show that, given $w_1, w_2 \in \Omega$, the integral of f from w_1 to w_2 is independent of the path. Let γ_0 and γ_1 be two curves from w_1 to w_2 with γ_j parametrized by $z_j(t)$, $t \in [0,1]$, $j = 0, 1$. Since Ω is simply connected, we can find a class of curves γ_s parametrized by $z_s(t)$, $t \in [0,1]$ for each $s \in [0,1]$ such that

$$z_s(0) = w_1 \quad \text{and} \quad z_s(1) = w_2 \quad \text{for all } s \in [0,1].$$

and the function $G(s,t) := z_s(t)$ is continuous on $[0,1] \times [0,1]$. One can see that the image $K = G([0,1] \times [0,1])$ is compact in Ω, there is $\varepsilon > 0$ such that $\operatorname{dist}(K, \partial\Omega) > 3\varepsilon$. By the uniform continuity of G, there is $\delta > 0$ such that

$$\sup_{t \in [0,1]} |z_{s_1}(t) - z_{s_2}(t)| < \varepsilon$$

provided $|s_1 - s_2| < \delta$. Therefore, for any $s_1, s_2 \in [0,1]$ with $|s_1 - s_2| < \delta$, we can find finitely many points $0 = t_0 < t_1 < ... < t_n = 1$ such that each closed curve l_k, $k = 1, ..., n$, consisting of the curve from $z_{s_1}(t_{k-1})$ to $z_{s_1}(t_k)$ along γ_{s_1}, the line segment from $z_{s_1}(t_k)$ to $z_{s_2}(t_k)$, the curve from $z_{s_2}(t_{k-1})$ to $z_{s_2}(t_k)$ along γ_{s_2}, and the line segment from $z_{s_2}(t_{k-1})$ to $z_{s_1}(t_{k-1})$, is contained in a disc with radius 2ε in Ω. By Theorem 10.3, we have

$$\int_{\gamma_{s_1}} f(z)dz - \int_{\gamma_{s_2}} f(z)dz = \sum_{k=1}^n \int_{l_k} f(z)dz = 0.$$

Dividing the interval $[0,1]$ into subintervals $[s_{k-1}, s_k]$ with length less than δ and repeating finitely many times of the above argument, the theorem is proved. □

Corollary 10.7. *If f is analytic on an open simply connected domain, then f has an antiderivative. And the integral of f from one point to another is independent of paths.*

Example 10.8. *Let γ be any closed curve lying in the disc $B_2(0)$. Then*

$$\int_\gamma \frac{\sin z}{(z^2+9)^5} dz = 0.$$

Example 10.9. *Let Ω be an open simply connected set with $1 \in \Omega$, $0 \notin \Omega$. Then there is a branch of the logarithm f on Ω such that*

$$f(x) = \ln x \quad \text{for } x \in \mathbb{R}, \quad x \text{ near } 1.$$

It can be done by defining

$$f(z) = \int_\gamma \frac{dw}{w}$$

for any curve γ in Ω from 1 to z. Notice that the integral of $\dfrac{1}{z}$ from 1 to z is independent of the choice of γ by Corollary 10.7. A similar argument in the proof of Theorem 9.5 gives $f'(z) = \dfrac{1}{z}$ on Ω, and hence

$$\left(ze^{-f(z)}\right)' = 0 \quad \text{on } \Omega.$$

Therefore, $ze^{-f(z)}$ is a constant. By taking the value at $z = 1$, we conclude that $ze^{-f(z)} \equiv 1$, that is,

$$e^{f(z)} = z \quad \text{on } \Omega.$$

As for $x \in \mathbb{R}$ near 1, we have

$$f(x) = \int_1^x \frac{dx}{x} = \ln x.$$

Example 10.10. *Let $\Omega \subset \mathbb{C}$ be an open simply connected set. If u is harmonic on $\{(x, y) : x + yi \in \Omega\}$, then there is an analytic function f on Ω such that $\operatorname{Re} f(z) = u(x, y)$ for $z = x + yi$. To see this, let*

$$g(z) = u_x(x, y) - iu_y(x, y) \quad \text{for } z = x + yi \in \Omega.$$

Then the real and imaginary parts of g are continuously differentiable and satisfy Cauchy-Riemann equations on the domain. Thus, g is analytic on Ω. By Corollary 10.7, g has an antiderivative F, that is,

$$F'(z) = g(z) = u_x(x, y) - iu_y(x, y) \quad \text{for } z = x + yi \in \Omega.$$

Let $\operatorname{Re} F(z) = w(x, y)$, by using the Cauchy-Riemann equations for F,

$$F'(z) = w_x(x, y) - iw_y(x, y) \quad \text{for } z = x + yi \in \Omega.$$

Therefore,

$$(w_x, w_y) = (u_x, u_y) \quad \text{for } x + yi \in \Omega,$$

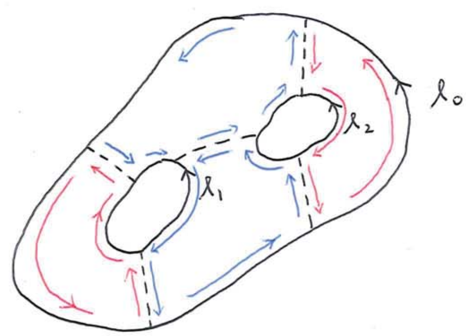
which implies that $w - u = c$ is a real constant. And hence $f = F - c$ is as desired.

For multiply connected domains, we have the following theorem.

Theorem 10.11. *Let l_0, l_1, ..., l_n be simple closed curves with counterclockwise orientation. l_k's, $k = 1, ..., n$, lying in the interior of l_0, are disjoint, whose interiors have no points in common. If f is analytic on all the curves and throughout the multiply connected domain consisting of the points inside l_0 and exterior to each l_k, $k = 1, ..., n$, then*

$$\int_{l_0} f(z)dz = \sum_{k=1}^{n} \int_{l_k} f(z)dz.$$

Proof. The theorem follows by dividing the domain into finitely many simply connected domains.



□

Corollary 10.12. *Let γ_1 and γ_2 be two simple closed curves with counterclockwise orientation. And γ_1 lies in the interior enclosed by γ_2. If f is analytic on the closed set consisting of γ_1, γ_2 and all points between them, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Example 10.13. *Let γ be a simple closed curve with counterclockwise orientation surrounding the origin. In order to evaluate the integral*

$$\int_{\gamma} \frac{dz}{z},$$

we notice that, as in Example 9.10,

$$\int_{C} \frac{dz}{z} = 2\pi i$$

for any circle C centered at the origin with counterclockwise orientation. Thus, by Corollary 10.12,

$$\int_{\gamma} \frac{dz}{z} = 2\pi i.$$

11 Cauchy Integral Formula

Theorem 11.1 (Cauchy integral formula)**.** *Let Ω is the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on $\overline{\Omega}$, the closure of Ω, then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

for all $z_0 \in \Omega$.

Proof. For $z_0 \in \Omega$, let C_ρ be the circle centered at z_0 with radius ρ sufficiently small such that $C_\rho \subset \Omega$. We assume that C_ρ is counterclockwise oriented. By Corollary 10.12,

$$\int_\gamma \frac{f(z)}{z - z_0} dz = \int_{C_\rho} \frac{f(z)}{z - z_0} dz.$$

Therefore, we have

$$\int_\gamma \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0} dz = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz. \tag{11.1}$$

As in Example 9.10,

$$\int_{C_\rho} \frac{1}{z - z_0} dz = 2\pi i.$$

Then (11.1) becomes

$$\int_\gamma \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

For the right-hand side on the last equality, we have

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \max_{z \in \overline{\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \cdot 2\pi\rho \longrightarrow 0 \quad \text{as } \rho \to 0.$$

Here $\max_{z \in \overline{\Omega}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$ is finite since f is analytic on $\overline{\Omega}$. Since ρ is arbitrary, we conclude that

$$\int_\gamma \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = 0.$$

The theorem then follows. □

Example 11.2. *Let $f(z) = \dfrac{\cos z}{z^2 + 9}$. To evaluate the integral*

$$\int_\gamma \frac{\cos z}{z\,(z^2 + 9)} dz,$$

where γ is the unit circle centered at the origin with counterclockwise orientation, we have

$$\int_\gamma \frac{\cos z}{z\,(z^2 + 9)} dz = \int_\gamma \frac{f(z)}{z - 0} dz = 2\pi i f(0) = \frac{2\pi i}{9}.$$

Theorem 11.3 (generalized Cauchy integral formula)**.** *Let Ω is the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on some open set containing $\overline{\Omega}$, the closure of Ω, then f is differentiable of all orders in Ω. Moreover,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$$

for all $z_0 \in \Omega$, $n \in \mathbb{N}\cup\{0\}$.

Proof. The proof is by induction on n. $n = 0$ is proved in Theorem 11.1. For $n \in \mathbb{N}$, suppose that f is $n-1$ times differentiable in Ω with

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n}}dz$$

for all $z_0 \in \Omega$. Then for each $z_0 \in \Omega$, $|h|$ small enough such that $z_0 + h \in \Omega$, we have

$$\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} = \frac{(n-1)!}{2\pi i}\int_\gamma f(z)\frac{1}{h}\left[\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n}\right]dz.$$

Notice that

$$\begin{aligned}\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} &= \left[\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right]\sum_{k=0}^{n-1}\frac{1}{(z-z_0-h)^k(z-z_0)^{n-k}}\\ &= \frac{h}{(z-z_0-h)(z-z_0)}\sum_{k=0}^{n-1}\frac{1}{(z-z_0-h)^k(z-z_0)^{n-k}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{h\to 0}\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} &= \frac{(n-1)!}{2\pi i}\int_\gamma f(z)\frac{1}{(z-z_0)^2}\frac{n}{(z-z_0)^{n-1}}dz\\ &= \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz,\end{aligned}$$

which completes the proof. □

Example 11.4. *Let $f(z) = e^{2z}$. To evaluate the integral*

$$\int_\gamma \frac{e^{2z}}{z^4}dz,$$

where γ is the unit circle centered at the origin with counterclockwise orientation, we have

$$\int_\gamma \frac{e^{2z}}{z^4}dz = \int_\gamma \frac{f(z)}{(z-0)^4}dz = \frac{2\pi i}{3!}f^{(3)}(0) = \frac{8\pi i}{3}.$$

Corollary 11.5. *If f is differentiable on an open set Ω, then f is differentiable of all orders on Ω. As a consequence, the real and imaginary parts of f are continuously differentiable of all orders.*

Proof. Given $z_0 \in \Omega$, there is a disc B centered at z_0 such that the closure of B is contained in Ω. Then we can apply Theorem 11.3 to conclude that f is differentiable of all orders in B. □

Remark 11.6. *Corollary 11.5 is used in the proof of Theorem 7.4.*

Theorem 11.7 (Morera's theorem)**.** *Let f is continuous on an open connected set Ω. If*

$$\int_\gamma f(z)dz = 0$$

for all closed curves in Ω, then f is differentiable on Ω.

Proof. By Theorem 9.5, f has an antiderivative F. By Corollary 11.5, F is differentiable of all orders on Ω. Hence, so is f. □

Corollary 11.8 (Cauchy's inequality)**.** *Suppose f is differentiable on an open set containing the closure of a disc B centered at z_0 with radius R. Let γ be the circle centered at z_0 with radius R, counterclockwise oriented, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \max_\gamma |f(z)|}{R^n}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof. By Cauchy integral formula,

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\
&\leq \frac{n!}{2\pi} \cdot \frac{\max_\gamma |f(z)|}{R^{n+1}} \cdot 2\pi R \\
&= \frac{n! \max_\gamma |f(z)|}{R^n}.
\end{aligned}$$

□

12 Liouville's Theorem and the Fundamental Theorem of Algebra

Theorem 12.1 (Liouville's theorem)**.** *If f is entire and bounded, then f is constant.*

Proof. Since f is bounded, there is a positive constant M such that

$$|f(z)| \leq M$$

for all $z \in \mathbb{C}$. By Cauchy's inequality,

$$|f'(z)| \leq \frac{M}{R}$$

for any $z \in \mathbb{C}$ and $R > 0$. Since R is arbitrary, we obtain that

$$f'(z) = 0 \quad \text{on } \mathbb{C}.$$

As a consequence, f is a constant on $\mathbb{C}$. □

Theorem 12.2 (fundamental theorem of algebra)**.** *Any non-constant polynomial has at least one root.*

Proof. Suppose on the contrary that there is a polynomial

$$P(z) = a_0 + a_1 z + ... + a_n z^n, \quad n \geq 1, \quad a_n \neq 0,$$

such that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $1/P(z)$ is entire. Now, we claim that $1/P(z)$ is bounded. Let

$$w(z) = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + ... + \frac{a_{n-1}}{z}, \quad z \neq 0.$$

By the triangle inequality,

$$|w(z)| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + ... + \frac{|a_{n-1}|}{|z|}.$$

By choosing $R > 0$ sufficiently large, we have

$$\frac{|a_k|}{R^{n-k}} \leq \frac{|a_n|}{2n} \quad \text{for } k = 0, 1, ..., n-1,$$

which gives

$$|w| \leq \frac{|a_n|}{2} \quad \text{for all } |z| \geq R.$$

Consequently,

$$|a_n + w| \geq |a_n| - |w| \geq \frac{|a_n|}{2} \quad \text{for all } |z| \geq R.$$

Then we have

$$|P(z)| = |a_n + w||z|^n \geq \frac{|a_n|}{2} R^n \quad \text{for all } |z| \geq R,$$

and hence

$$\left| \frac{1}{P(z)} \right| \leq \frac{2}{|a_n| R^n} \quad \text{for all } |z| \geq R. \tag{12.1}$$

Since $1/P(z)$ is continuous on the set $\{|z| \leq R\}$, it is bounded on $\{|z| \leq R\}$. Therefore, $1/P(z)$ is entire and bounded. By Liouville's theorem, $1/P(z)$ is a constant on $\mathbb{C}$, which leads a contradiction. $\square$

Corollary 12.3. *A polynomial P of order n, $n \geq 1$ has precisely n roots in $\mathbb{C}$. P can be expressed as*

$$P(z) = c(z - z_1)(z - z_2)...(z - z_n),$$

where c, z_1, ..., z_n are constants with $c \neq 0$.

Proof. For

$$P(z) = a_0 + a_1 z + ... + a_n z^n,$$

by Theorem 12.2, there is a root z_1 of P. We have

$$\begin{aligned} P(z) &= P((z - z_1) + z_1) \\ &= a_0 + a_1((z - z_1) + z_1) + ... + a_n((z - z_1) + z_1)^n \\ &= b_1(z - z_1) + b_2(z - z_1)^2 + ... + b_n(z - z_1)^n \end{aligned}$$

for some $b_1, ..., b_n \in \mathbb{C}$, and $b_n = a_n$. Thus,

$$\begin{aligned} P(z) &= (z - z_1)\left[b_1 + b_2(z - z_1) + ... + b_n(z - z_1)^{n-1}\right] \\ &= (z - z_1)Q(z), \end{aligned}$$

where Q is a polynomial of order $n - 1$. By Theorem 12.2 again, there is a root z_2 of Q. We then prove the corollary inductively. □

13 Maximum Modulus Principle

Theorem 13.1 (maximum modulus principle)**.** *If f is non-constant and analytic on an open connected set Ω, then there is no point $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$.*

Lemma 13.2. *If $|f(z)| \leq |f(z_0)|$ for all $z \in B_R(z_0)$, then $f(z) = f(z_0)$ for all $z \in B_R(z_0)$.*

Proof. Let C_ρ be the circle centered at z_0 with radius $\rho \in (0, R)$ and counterclockwise oriented. By Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z - z_0} dz$$

for all $\rho \in (0, R)$. Then

$$|f(z_0)| \leq \frac{1}{2\pi} \max_{C_\rho} \frac{|f(z)|}{|z - z_0|} \cdot 2\pi\rho \leq \frac{1}{2\pi} \cdot \frac{|f(z_0)|}{\rho} \cdot 2\pi\rho = |f(z_0)|$$

for all $\rho \in (0, R)$. Thus both the inequalities above are equalities, which implies that

$$|f(z)| = |f(z_0)| \quad \text{on } C_\rho.$$

Since $\rho \in (0, R)$ is arbitrary, $|f(z)| = |f(z_0)|$ on $B_R(z_0)$. By Example 6.20, f is a constant on $B_R(z_0)$, which completes the proof. □

Proof of Theorem 13.1. Suppose on the contrary that there is $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. For any $w \in \Omega$, there is a polygonal line L connecting z_0 and w. Let $0 < \delta < \text{dist}(L, \partial\Omega)$, L can be covered by finitely many discs $B_\delta(z_k)$, $z_k \in L$, $k = 0, 1, ..., N$, and $w = z_N$. Moreover, $z_k \in B_\delta(z_{k-1})$ for each $k = 1, 2, ..., N$. By Lemma 13.2, f is a constant on $B_\delta(z_0)$. Thus $f(z_1) = f(z_0)$, and hence $|f(z)| \leq |f(z_1)|$ on $B_\delta(z_1)$. Continue in this manner, we conclude that $f(w) = f(z_0)$. That is, f is constant on Ω, which leads a contradiction. □

Remark 13.3.

(i) *Under the assumptions of Theorem 13.1, if f is continuous on the closure of Ω, then the maximum value of $|f(z)|$ on the closure of Ω must occur on the boundary of Ω.*

(ii) *Applying Theorem 13.1 to $1/f(z)$, the minimum of $|f(z)|$ cannot be obtained at an interior point of Ω provided that $f(z) \neq 0$ for all $z \in \Omega$.*

(iii) *Applying Theorem 13.1 to functions $e^{f(z)}$ and $e^{-if(z)}$, similar properties hold for the real and imaginary parts of f as in Theorem 13.1.*

Example 13.4. *We can use the maximum modulus principle to prove the fundamental theorem of algebra. Suppose that P is a non-constant polynomial of order n and has no root on $\mathbb{C}$. Then $1/P(z)$ is analytic on $B_R(0)$ for all $R > 0$. As in (12.1), we have*

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n|R^n} \quad \text{on the circle } \{z : |z| = R\}$$

provided R sufficiently large. By the maximum modulus principle,

$$\left|\frac{1}{P(z)}\right| \leq \frac{2}{|a_n|R^n} \quad \text{on } B_R(0).$$

Taking $R \to \infty$, we conclude that $\dfrac{1}{P(z)} = 0$ on $\mathbb{C}$, a contradiction.

Example 13.5. *Let $f(z) = (z+1)^2$ be defined on the closed triangle T with vertices $z = 0$, $z = 2$ and $z = i$. Notice that $|f(z)|$ can be interpreted as the square of the distance between -1 and $z \in T$. The maximum and minimum values of $|f(z)|$ occur at $z = 2$ and $z = 0$, respectively.*

14 Taylor Series and Laurent Series

Definition 14.1. *For $z_n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, the series $\displaystyle\sum_{n=0}^{\infty} z_n$ converges to the sum z if the partial sum*

$$\sum_{n=0}^{N} z_n \longrightarrow z \quad \text{as } N \to \infty.$$

If it does not converge, we say that it diverges. And we say that the series $\displaystyle\sum_{n=0}^{\infty} z_n$ converges absolutely if the series $\displaystyle\sum_{n=0}^{\infty} |z_n|$ converges.

Proposition 14.2. *Absolute convergence implies convergence.*

Proposition 14.3. *Given a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that the series converges absolutely if $|z| < R$ and diverges if $|z| > R$. Moreover, R is given by*

$$R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}.$$

Definition 14.4. *R given in Proposition 14.3 is called the radius of convergence of the power series. And $B_R(0)$ is called the disc of convergence.*

Proof. For $|z| < R$, there is $\varepsilon > 0$ small enough such that

$$\left(R^{-1} + \varepsilon\right)|z| = r < 1.$$

By the definition of R,

$$|a_n|^{1/n} \leq R^{-1} + \varepsilon$$

for all n large, which gives

$$|a_n||z|^n \leq \left(R^{-1} + \varepsilon\right)^n |z|^n = r^n.$$

By a comparison with the series $\displaystyle\sum_{n=0}^{\infty} r^n$, the series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

If $|z| > R$,

$$\limsup_{n\to\infty} |a_n z^n| \geq \limsup_{n\to\infty} R^{-n}|z|^n = \infty.$$

Thus, the series cannot converge for $|z| > R$. □

Theorem 14.5. *A function defined by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C},$$

with positive radius of convergence, is differentiable on its disc of convergence. And its derivative can be represented by the power series

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^n,$$

which has the same radius of convergence as f.

Proof. Let

$$g(z) = \sum_{n=1}^{\infty} n a_n z^n.$$

Since

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |n a_n|^{1/n},$$

g has the same radius of convergence as f. Let R be the radius of convergence of f, and divide f into

$$f(z) = S_N(z) + R_N(z),$$

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad R_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

For $|z_0| < r < R$, $|h|$ sufficiently small such that $|z_0 + h| < r$, we have

$$\begin{aligned}
&\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \\
&= \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right) + (S_N'(z_0) - g(z_0)) + \left(\frac{R_N(z_0+h) - R_N(z_0)}{h} \right)
\end{aligned}$$

Given $\varepsilon > 0$, since

$$\left| \frac{R_N(z_0+h) - R_N(z_0)}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z_0+h)^n - z_0^n}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

there is $N_1 \in \mathbb{N}$ sufficiently large such that

$$\left| \frac{R_N(z_0+h) - R_N(z_0)}{h} \right| < \frac{\varepsilon}{3}$$

for all h with $|z_0 + h| < r$ and $N \geq N_1$. Also, since

$$\lim_{N \to \infty} S_N'(z_0) = g(z_0),$$

there is $N_2 \in \mathbb{N}$ sufficiently large such that

$$|S_N'(z_0) - g(z_0)| < \frac{\varepsilon}{3}$$

for all h with $|z_0 + h| < r$ and $N \geq N_2$. Now, we fix $N \geq \max\{N_1, N_2\}$, there is $\delta > 0$ such that

$$\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3}$$

provided $|h| < \delta$. Therefore,

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \right| < \varepsilon$$

provided $|h| < \delta$, that is,

$$f'(z_0) = g(z_0).$$

□

Corollary 14.6. *A function defined by a power series with positive radius of convergence is infinitely many times differentiable on its disc of convergence. And all the higher derivatives can be represented by the power series obtained by termwise differentiation and have the same radius of convergence as f.*

Theorem 14.7. *Suppose that f is analytic on a disc $B_R(z_0)$. Then f can be represented as*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in B_R(z_0), \tag{14.1}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

Remark 14.8.

(i) (14.1) is called the Taylor series of f about z_0. In particular, if $z_0 = 0$, it is called the Maclaurin series of f.

(ii) The coefficients of Taylor series are unique.

Proof. Without loss of generality, we may assume that $z_0 = 0$. By Cauchy integral formula, for any $z \in B_R(0)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw, \tag{14.2}$$

where γ is the circle centered at 0 with radius $\dfrac{|z| + R}{2}$ and counterclockwise orientation. Notice that

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - z/w} = \frac{1}{w} \left[\sum_{n=0}^{N} \left(\frac{z}{w}\right)^n + \frac{1}{1 - z/w} \left(\frac{z}{w}\right)^{N+1} \right].$$

Thus, (14.2) becomes

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w} \sum_{n=0}^{N} \left(\frac{z}{w}\right)^n dw + \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w} \frac{1}{1 - z/w} \left(\frac{z}{w}\right)^{N+1} dw \\ &= \sum_{n=0}^{N} \frac{z^n}{2\pi i} \int_\gamma \frac{f(w)}{w^{n+1}} dw + \frac{z^{N+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w - z) w^{N+1}} dw. \end{aligned} \tag{14.3}$$

By the generalized Cauchy integral formula,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{n+1}} dw = \frac{f^{(n)}(0)}{n!}, \quad n = 0, 1, ..., N.$$

Thus, (14.3) reduces to

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n + \frac{z^{N+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w - z) w^{N+1}} dw.$$

Now, let $M = \max_\gamma |f|$, then

$$\begin{aligned} \left| \frac{z^{N+1}}{2\pi i} \int_\gamma \frac{f(w)}{(w - z) w^{N+1}} dw \right| &\le \frac{|z|^{N+1}}{2\pi} \cdot \frac{M}{\dfrac{R - |z|}{2} \left(\dfrac{R + |z|}{2}\right)^{N+1}} \cdot 2\pi \left(\frac{R + |z|}{2}\right) \\ &= M \cdot \frac{R + |z|}{R - |z|} \left(\frac{2|z|}{R + |z|}\right)^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty. \end{aligned}$$

Therefore, we complete the proof. $\square$

Example 14.9. *Let* $f(z) = \dfrac{1}{1-z}$*. We have*

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad z \neq 1, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{on } B_1(0).$$

As for the Taylor series of f *about* i*, we have*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad \text{on } B_{\sqrt{2}}(i).$$

Example 14.10. *Let* $f(z) = e^z$*. We have*

$$f^{(n)}(z) = e^z \quad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{on } \mathbb{C}.$$

We can use the Taylor series of e^z *to show that*

$$e^{2z} = \sum_{n=0}^{\infty} \frac{2^n z^n}{n!} \quad \text{on } \mathbb{C}.$$

Moreover,

$$z^3 e^{2z} = \sum_{n=0}^{\infty} \frac{2^n z^{n+3}}{n!} \quad \text{on } \mathbb{C}.$$

Example 14.11. *Let* $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$*. We have*

$$f^{(n)}(z) = \frac{i^n e^{iz} - (-i)^n e^{-iz}}{2i} \quad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\sin z = \sum_{n=0}^{\infty} \frac{i^n - (-i)^n}{2i} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \quad \text{on } \mathbb{C}. \tag{14.4}$$

Example 14.12. *Let* $f(z) = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$*. We have*

$$f^{(n)}(z) = \frac{i^n e^{iz} + (-i)^n e^{-iz}}{2} \quad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\cos z = \sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \quad \text{on } \mathbb{C}.$$

The Taylor series of $\cos z$ *can also be obtained by differentiating (14.4) term by term*

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot (2k+1) z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \quad \text{on } \mathbb{C}.$$

Example 14.13. *Let* $f(z) = \sinh z = \dfrac{e^z - e^{-z}}{2}$. *We have*

$$f^{(n)}(z) = \frac{e^z - (-1)^n e^{-z}}{2} \quad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\sinh z = \sum_{n=0}^{\infty} \frac{1 - (-1)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \quad \text{on } \mathbb{C}.$$

Example 14.14. *Let* $f(z) = \cosh z = \dfrac{e^z + e^{-z}}{2}$. *We have*

$$f^{(n)}(z) = \frac{e^z + (-1)^n e^{-z}}{2} \quad \text{on } \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$\cosh z = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} \cdot \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \quad \text{on } \mathbb{C}.$$

Theorem 14.15. *Suppose that* f *is analytic on an annulus* $B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}$. *Then* f *can be represented as*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad z \in B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}, \tag{14.5}$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

with any simple closed curve γ *in* $B_{R_2}(z_0) \backslash \overline{B_{R_1}(z_0)}$ *around* z_0 *with counterclockwise orientation.*

Remark 14.16.

(i) (14.5) is called the Laurent series of f *about* z_0.

(ii) If f *is also analytic on* $\overline{B_{R_1}(z_0)}$, *then the Taylor series of* f *and the Laurent series of* f *about* z_0 *agree with each other. In fact,* $c_n = 0$ *for all* $n < 0$ *by Cauchy-Goursat theorem. Moreover, by Cauchy integral formula,* $c_n = a_n$ *for* $n \in \mathbb{N} \cup \{0\}$, *where* a_n *are the Taylor coefficients of* f.

(iii) The coefficients of Laurent series are unique. Suppose that

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=-\infty}^{\infty} \tilde{c}_n(z-z_0)^n \quad \text{on } B_{R_2}(z_0)\backslash\overline{B_{R_1}(z_0)}.$$

Then, for $m \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^{n-m-1} = \sum_{n=-\infty}^{\infty} \tilde{c}_n(z-z_0)^{n-m-1}.$$

Since

$$\int_\gamma (z-z_0)^{n-m-1} = \begin{cases} 2\pi i & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}$$

for any simple closed curve γ around z_0 with counterclockwise orientation, then we conclude

$$c_m = \tilde{c}_m, \quad m \in \mathbb{Z}.$$

Here the interchange of orders of summation and multiplication or integration can be justified by absolute convergence.

Proof. Without loss of generality, we may assume that $z_0 = 0$. For $z \in B_{R_2}(0)\backslash\overline{B_{R_1}(0)}$, let γ_1 and γ_2 be the circles centered at 0 with radius r_1 and r_2, respectively, such that $R_1 < r_1 < |z| < r_2 < R_2$. There is $\varepsilon > 0$ sufficiently small such that the closed disk $\overline{B_\varepsilon(z)}$ is contained in the annulus $B_{r_2}(0)\backslash\overline{B_{r_1}(0)}$. Let γ be the boundary of $B_\varepsilon(z)$. We assume that γ, γ_1 and γ_2 are all counterclockwise oriented. By Theorem 10.11,

$$\int_{\gamma_2} \frac{f(w)}{w-z}dw - \int_{\gamma_1} \frac{f(w)}{w-z}dw - \int_{\gamma} \frac{f(w)}{w-z}dw = 0. \tag{14.6}$$

By the Cauchy integral formula,

$$\int_\gamma \frac{f(w)}{w-z}dw = 2\pi i f(z).$$

Put this into (14.6), we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{z-w}dw. \tag{14.7}$$

Notice that

$$\frac{1}{w-z} = \frac{1}{w}\cdot\frac{1}{1-z/w} = \frac{1}{w}\left[\sum_{n=0}^{N}\left(\frac{z}{w}\right)^n + \frac{1}{1-z/w}\left(\frac{z}{w}\right)^{N+1}\right]$$

for $w \in \gamma_2$. Similarly,

$$\frac{1}{z-w} = \frac{1}{z}\left[\sum_{n=0}^{N}\left(\frac{w}{z}\right)^n + \frac{1}{1-w/z}\left(\frac{w}{z}\right)^{N+1}\right]$$

for $w \in \gamma_1$. Then (14.7) becomes

$$f(z) = \sum_{n=0}^{N} \frac{z^n}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w^{n+1}} dw + \frac{z^{N+1}}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)w^{N+1}} dw$$
$$+ \sum_{n=0}^{N} \frac{z^{-(n+1)}}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w^{-n}} dw + \frac{z^{-(N+1)}}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-w)w^{-(N+1)}} dw.$$

Now, let $M = \max_{\gamma_1 \cup \gamma_2} |f|$. Then

$$\begin{aligned} \left| \frac{z^{N+1}}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)w^{N+1}} dw \right| &\leq \frac{|z|^{N+1}}{2\pi} \cdot \frac{M}{(r_2 - |z|){r_2}^{N+1}} \cdot 2\pi r_2 \\ &= M \cdot \frac{r_2}{r_2 - |z|} \left(\frac{|z|}{r_2} \right)^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{z^{-(N+1)}}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-w)w^{-(N+1)}} dw \right| &\leq \frac{|z|^{-(N+1)}}{2\pi} \cdot \frac{M}{(|z| - r_1){r_1}^{-(N+1)}} \cdot 2\pi r_1 \\ &= M \cdot \frac{r_1}{|z| - r_1} \left(\frac{r_1}{|z|} \right)^{N+1} \longrightarrow 0 \quad \text{as } N \to \infty. \end{aligned}$$

Therefore, we complete the proof. □

Example 14.17. *Let* $f(z) = \dfrac{1}{z(1+z^2)}$. *Since*

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1.$$

Therefore,

$$\frac{1}{z(1+z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}, \quad 0 < |z| < 1,$$

is the Laurent series of f.

Example 14.18. *Let* $f(z) = \dfrac{z+1}{z-1}$. *For* $|z| < 1$,

$$\frac{z+1}{z-1} = -z \cdot \frac{1}{1-z} - \frac{1}{1-z} = -z \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -1 - 2 \sum_{n=1}^{\infty} z^n,$$

which is the Taylor series of f. *And for* $|z| > 1$,

$$\frac{z+1}{z-1} = \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} = \left(1 + \frac{1}{z}\right) \sum_{n=0}^{\infty} \frac{1}{z^n} = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{z^n},$$

which is the Laurent series of f.

Example 14.19. *Notice that*

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!},$$

which is the Laurent series of $e^{1/z}$. Let γ be the circle centered at 0 with radius R, counter-clockwise oriented. Then the coefficients of Laurent series

$$c_{-n} = \frac{1}{2\pi i}\int_{\gamma} \frac{e^{1/z}}{z^{-n+1}} dz = \frac{1}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

It can be used to evaluate the integrals

$$\int_{\gamma} \frac{e^{1/z}}{z^{-n+1}} dz = \frac{2\pi i}{n!}, \quad n \in \mathbb{N} \cup \{0\}.$$

For the sake of completion, we list some results used in this section as follows.

Proposition 14.20. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1},$$

and g is a bounded and continuous function on $\Omega \subset B_R(z_0)$, then

$$f(z)g(z) = \sum_{n=0}^{\infty} a_n g(z)(z - z_0)^n \quad \text{on } \Omega.$$

Moreover, if γ is a curve in Ω, then

$$\int_{\gamma} f(z)g(z)dz = \sum_{n=0}^{\infty} \int_{\gamma} a_n g(z)(z - z_0)^n dz.$$

Lemma 14.21. *If $R_0 < R$, the series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

absolutely converges uniformly on $B_{R_0}(z_0)$.

Proof. The proof is similar to the proof of Proposition 14.3. There is $\varepsilon > 0$ small enough such that

$$\left(R^{-1} + \varepsilon\right) R_0 = r < 1.$$

By the definition of R, for any $z \in B_{R_0}(z_0)$,

$$|a_n||z - z_0|^n \leq \left(R^{-1} + \varepsilon\right)^n R_0^n = r^n,$$

which gives

$$\sum_{n=N+1}^{\infty} |a_n||z - z_0|^n \leq \sum_{n=N+1}^{\infty} r^n \leq \frac{r^{N+1}}{1 - r}.$$

□

Proof of Proposition 14.20. For each $z \in B_R(z_0)$, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ absolutely converges. In particular, for $z \in \Omega$,

$$\sum_{n=N+1}^{\infty} |a_n g(z)(z-z_0)^n| \leq \sup_{\Omega} |g| \cdot \sum_{n=N+1}^{\infty} |a_n||z-z_0|^n \longrightarrow 0$$

as $N \to \infty$. Since $\gamma \in \Omega \subset B_R(z_0)$, there is $R_0 < R$ such that $\gamma \in B_{R_0}(z_0)$. Thus, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ absolutely converges uniformly on γ. We have

$$\left| \int_\gamma g(z) \sum_{n=N+1}^{\infty} a_n(z-z_0)^n dz \right| \leq \sup_\gamma |g| \cdot \sup_\gamma \left| \sum_{n=N+1}^{\infty} a_n(z-z_0)^n \right| \cdot \text{length}(\gamma) \longrightarrow 0$$

as $N \to \infty$. $\square$

Corollary 14.22. *If*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad R_1 < |z-z_0| < R_2,$$

where

$$R_1 = \limsup_{n\to\infty} |c_{-n}|^{1/n}, \quad \textit{and} \quad R_2 = \left(\limsup_{n\to\infty} |c_n|^{1/n} \right)^{-1},$$

and g is a bounded and continuous function on $\Omega \subset \{z \in \mathbb{C} : R_1 < |z| < R_2\}$, then

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} c_n g(z)(z-z_0)^n \quad \text{on } \Omega.$$

Moreover, if γ is a curve in Ω, then

$$\int_\gamma f(z)g(z)dz = \sum_{n=-\infty}^{\infty} \int_\gamma c_n g(z)(z-z_0)^n dz.$$

15 Isolated Singularities

In this section, we assume that f is analytic on a punctured disc $\{z \in \mathbb{C} : 0 < |z-z_0| < R\}$ for some $R > 0$. Since f is analytic on annulus $B_R(z_0)\backslash\overline{B_r(z_0)}$ for any $r \in (0, R)$, f can be represented as Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad r < |z-z_0| < R,$$

where $c_n \in \mathbb{C}$, $n \in \mathbb{Z}$. By the uniqueness of the coefficients of Laurent series, c_n's are independent of r, and hence

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad 0 < |z-z_0| < R. \tag{15.1}$$

There are three types of singularities: removable singularities, poles, and essential singularities.

Definition 15.1. *Let f be analytic on a punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ with representation (15.1).*

(i) If $c_n = 0$ for all $n < 0$, then z_0 is called a removable singularity.

(ii) If there is $m \in \mathbb{N}$ such that $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$, then z_0 is called a pole of order m.

(iii) If there are infinitely many $c_n \neq 0$ with $n < 0$, then z_0 is called an essential singularity.

Example 15.2.

(i) Let $f(z) = \dfrac{1}{z(1+z^2)}$ be defined on $B_1(0)\backslash\{0\}$, then 0 is a pole of f.

(ii) Let $f(z) = e^{1/z}$ be defined on $B_1(0)\backslash\{0\}$, then 0 is an essential singularity of f.

Proposition 15.3. *If z_0 is a removable singularity of f, by defining*

$$g(z) = \begin{cases} f(z), & 0 < |z - z_0| < R, \\ c_0, & z = z_0, \end{cases}$$

then the function g is analytic on $B_R(z_0)$.

Proof. Notice that

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has the radius of convergence not less than R, otherwise f cannot be defined on $B_R(z_0)\backslash\{z_0\}$. By Theorem 14.5, g is analytic on $B_R(z_0)$. □

Proposition 15.4. *If f is bounded and analytic on $B_R(z_0)\backslash\{z_0\}$. Then z_0 is a removable singularity of f.*

Proof. Notice that f can be represented as the Laurent series (15.1). In fact, by Theorem 14.15,

$$c_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(w)}{(w - z_0)^{n+1}} dw, \quad n \in \mathbb{Z},$$

where γ_ρ is the circle centered at z_0 with radius ρ and counterclockwise orientation, for any $\rho > 0$. For each $n < 0$,

$$|c_n| \leq \frac{1}{2\pi} \cdot \frac{\sup |f|}{\rho^{n+1}} \cdot 2\pi\rho = \frac{\sup |f|}{\rho^n}.$$

Since ρ is arbitrary, we conclude that $c_n = 0$ for all $n < 0$. That is, z_0 is a removable singularity. □

Proposition 15.5. *If z_0 is a pole of f, then*

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Proof. If z_0 is a pole of order m, $m \in \mathbb{N}$, we have

$$f(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n, \quad 0 < |z - z_0| < R,$$

where $c_{-m} \neq 0$. By letting

$$g(z) = (z - z_0)^m f(z) = \sum_{n=0}^{\infty} c_{n-m} (z - z_0)^n,$$

which is analytic on $B_R(z_0) \backslash \{z_0\}$, then z_0 is a removable singularity of g with

$$\lim_{z \to z_0} g(z) = c_{-m} \neq 0.$$

Therefore,

$$|f(z)| = \frac{|g(z)|}{|z - z_0|^{-m}} \longrightarrow \infty \quad \text{as } z \to z_0.$$

□

Proposition 15.6. *If z_0 is an essential singularity of f, then for any $c \in \mathbb{C}$, there is a sequence $z_k \to z_0$ such that*

$$|f(z_k) - c| \longrightarrow 0 \quad \text{as } k \to \infty.$$

Proof. Suppose on the contrary that there are $c \in \mathbb{C}$, $\varepsilon_0 > 0$ and $\delta_0 \in (0, R)$ such that

$$|f(z) - c| > \varepsilon_0 \quad \text{for all } z \in B_{\delta_0}(z_0) \backslash \{z_0\}.$$

We define

$$g(z) = \frac{1}{f(z) - c} \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}.$$

Then g is analytic and

$$|g(z)| \leq \frac{1}{|f(z) - c|} \leq \frac{1}{\varepsilon_0} \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}.$$

By Proposition 15.4, z_0 is a removable singularity of g. That is,

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{on } B_{\delta_0}(z_0) \backslash \{z_0\}$$

for some $a_n \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$. If $a_0 \neq 0$, we have

$$\lim_{z \to z_0} g(z) = a_0 \neq 0,$$

which implies that

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{1}{g(z)} + c = \frac{1}{a_0} + c.$$

Therefore, f is bounded on $B_\sigma(z_0)\backslash\{z_0\}$ for some $\sigma > 0$. By Proposition 15.4, z_0 is a removable singularity of f, a contradiction. We must have $a_0 = 0$. Let N be the least number (if it exists) such that

$$a_0 = a_1 = ... = a_{N-1} = 0 \quad \text{and} \quad a_N \neq 0.$$

Define

$$h(z) = \sum_{n=0}^{\infty} a_{n+N}(z - z_0)^n \quad \text{on } B_{\delta_0}(z_0),$$

which is analytic with $h(z_0) = a_N \neq 0$. Then

$$g(z) = (z - z_0)^N \sum_{n=N}^{\infty} a_n(z - z_0)^{n-N} = (z - z_0)^N h(z) \quad \text{on } B_{\delta_0}(z_0)\backslash\{z_0\}.$$

Moreover, on $B_{\delta_0}(z_0)\backslash\{z_0\}$,

$$f(z) = \frac{1}{g(z)} + c = \frac{1}{h(z)(z - z_0)^N} + c,$$

which implies that z_0 is a pole of order N of f, again a contradiction. As a consequence, $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, which implies that

$$g(z) = 0 \quad \text{on } B_{\delta_0}(z_0)\backslash\{z_0\}.$$

It is still impossible. We then complete the proof. $\square$

Theorem 15.7 (Picard's theorem)**.** *If z_0 is an essential singularity of f, then on any punctured neighborhood of z_0, f takes all complex values, with at most one exception, infinitely often.*

Example 15.8. *Let $f(z) = e^{1/z}$. Then 0 is an essential singularity of f. The value 0 is the only exceptional value which cannot be taken by f on any punctured neighborhood of the point 0. For any non-zero complex value $c = \rho e^{i\theta}$, we solve the equation*

$$e^{1/z} = e^{\frac{1}{|z|^2}(x-iy)} = \rho e^{i\theta} = c, \quad z = x + yi. \tag{15.2}$$

We have

$$e^{x/|z|^2} = \rho \quad \text{and} \quad e^{-iy/|z|^2} = e^{i\theta}.$$

That is,

$$\frac{x}{|z|^2} = \ln \rho$$

and

$$\frac{y}{|z|^2} = -\theta + 2n\pi, \quad n \in \mathbb{Z}.$$

The last two equations imply

$$\frac{1}{|z|^2} = (\ln \rho)^2 + (-\theta + 2n\pi)^2. \tag{15.3}$$

Thus, we know that for each $n \in \mathbb{Z}$, $z_n = x_n + y_n i$ is a solution of (15.2), where

$$x_n = \frac{\ln \rho}{(\ln \rho)^2 + (-\theta + 2n\pi)^2} \quad \text{and} \quad y_n = \frac{-\theta + 2n\pi}{(\ln \rho)^2 + (-\theta + 2n\pi)^2}.$$

By (15.3), $z_n \to 0$ as $n \to \infty$. That is, c can be taken by f infinitely many times on any punctured neighborhood of 0.

16 Isolation of Points in Preimage

Definition 16.1. *Let f be defined on an open connected set Ω. The image of a set $X \subset \Omega$ under f is defined by*

$$f(X) = \{w \in \mathbb{C} : w = f(z) \text{ for some } z \in X\}.$$

The preimage of a set $Y \subset f(\Omega)$ under f is defined by

$$f^{-1}(Y) = \{z \in \mathbb{C} : f(z) = w \text{ for some } w \in Y\}.$$

If $Y = \{c\}$, we may write $f^{-1}(Y) = f^{-1}(c)$ for simplicity.

Proposition 16.2. *If f is non-constant, analytic, and $f(z_0) = c$ for some $z_0 \in \Omega$, then there is $\varepsilon > 0$ such that*

$$B_\varepsilon(z_0) \cap f^{-1}(c) = \{z_0\}.$$

Proof. Near z_0, f has the Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} a_n(z - z_0)^n = c + \sum_{n=1}^{\infty} a_n(z - z_0)^n.$$

Since f is non-constant, there is a smallest $N \in \mathbb{N}$ such that $a_N \neq 0$. And we can rewrite above representation as

$$f(z) = c + \sum_{n=N}^{\infty} a_n(z - z_0)^n = c + (z - z_0)^N g(z),$$

where

$$g(z) = \sum_{n=0}^{\infty} a_{N+n}(z - z_0)^n.$$

Since $a_N \neq 0$, there is $\varepsilon > 0$ such that

$$g(z) \neq 0 \quad \text{on } B_\varepsilon(z_0),$$

which implies that

$$f(z) \neq c \quad \text{on } B_\varepsilon(z_0) \backslash \{z_0\}.$$

□

Theorem 16.3. *Suppose that f is analytic on an open connected set Ω. If $f(z_n) = 0$, where $z_n \in \Omega$ is a sequence of distinct points with a limit point in Ω, then f is identical to 0.*

Proof. By taking a subsequence, still indexed by n, $z_0 = \lim_{n\to\infty} z_n$. By the continuity of f, $f(z_0) = 0$. Suppose that f is not identical to 0, by Proposition 16.2, there is $\varepsilon > 0$ such that

$$B_\varepsilon(z_0) \cap f^{-1}(0) = \{z_0\},$$

a contradiction. □

Corollary 16.4. *Suppose that f and g are analytic on an open connected set Ω. If $f(z_n) = g(z_n)$, where $z_n \in \Omega$ is a sequence of distinct points with a limit point in Ω, then f is identical to g.*

Remark 16.5. *If f and F are analytic on Ω' and Ω, respectively, where $\Omega' \subset \Omega$. If $f(z) = F(z)$ on Ω', then F is an analytic continuation of f. Corollary 16.4 guarantees that there can be only one such analytic continuation. In particular, suppose that f_1 and f_2 are analytic on Ω_1 and Ω_2, respectively, and $f_1 = f_2$ on $\Omega_1 \cap \Omega_2 \neq \phi$. Then the function*

$$g(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1 \backslash \Omega_2, \\ f_2(z) & \text{if } z \in \Omega_2, \end{cases}$$

on $\Omega_1 \cup \Omega_2$ is an analytic continuation of both f_1 and f_2.

Theorem 16.6 (reflection principle)**.** *Let Ω be an open connected set which is symmetric with respect to the real axis. $\Omega = \Omega^+ \cup I \cup \Omega^-$, where Ω^+ is the upper half part, Ω^- is the lower half part, and $I = \Omega \cap \mathbb{R}$. If f is analytic on Ω, then*

$$\overline{f(z)} = f(\overline{z}), \quad z \in \Omega, \tag{16.1}$$

if and only if f is real-valued on I.

Proof. Suppose that (16.1) holds. For $x \in I$, we have

$$\overline{f(x)} = f(\overline{x}) = f(x),$$

which gives $f(x) \in \mathbb{R}$.

On the other hand, if f is real-valued on I, we can define

$$g(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup I, \\ \\ \overline{f(\overline{z})} & \text{if } z \in \Omega^-. \end{cases}$$

Then g is analytic on Ω^+. For each $z_0 \in \Omega^-$, we have $\overline{z_0} \in \Omega^+$, and hence

$$g(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n$$

in a neighborhood of $\overline{z_0}$ in Ω^+. By the definition of g,

$$g(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n$$

in a neighborhood of z_0 in Ω^-. That is, g is analytic on Ω^-. And for each $x_0 \in I$, we have

$$f(z) = \sum_{n=0}^{\infty} b_n (z - x_0)^n$$

in a neighborhood of x_0, say $B_\delta(x_0)$. In addition, b_n's are all real since f takes real values on I. Hence,

$$g(z) = \sum_{n=0}^{\infty} b_n (z - x_0)^n \quad \text{on } B_\delta(x_0) \cap \left(\Omega^+ \cup I\right).$$

Moreover, for $z \in B_\delta(x_0) \cap \Omega^-$,

$$g(z) = \sum_{n=0}^{\infty} \overline{b_n}(z - \overline{x_0})^n = \sum_{n=0}^{\infty} b_n(z - x_0)^n.$$

We conclude that

$$g(z) = \sum_{n=0}^{\infty} b_n(z - x_0)^n \quad \text{on } B_\delta(x_0).$$

Therefore g is also analytic on I, and hence analytic on Ω. Since $f = g$ on Ω^+, by Corollary 16.4, f is identical to g on Ω. That is, (16.1) holds. □

17 Residue Theorem

Definition 17.1. *If f is analytic and has the Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$$

on a punctured neighborhood $B_\sigma(z_0)\backslash\{z_0\}$, $\sigma > 0$. Then the coefficient c_{-1} is called the residue of f at z_0, denoted by

$$\operatorname{Res}(f; z_0) = c_{-1}.$$

Proposition 17.2. *Let γ be the circle centered at z_0 with radius R and counterclockwise orientation. If f is analytic on $\overline{B_R(z_0)}\backslash\{z_0\}$, $z_0 \in \Omega$, then*

$$\int_\gamma f(z)dz = 2\pi i \operatorname{Res}(f; z_0).$$

Proof. Recall that the coefficient of Laurent series

$$c_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)dz.$$

Equivalently,

$$\int_\gamma f(z)dz = 2\pi i\, c_{-1} = 2\pi i \operatorname{Res}(f; z_0).$$

□

Corollary 17.3 (residue theorem)**.** *Let Ω be the open set enclosed by a simple closed curve γ with counterclockwise orientation. If f is analytic on $\overline{\Omega}\backslash\{z_1, ..., z_N\}$ for N distinct points $z_1, ..., z_N \in \Omega$, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f; z_k).$$

Proof. By choosing $\delta > 0$ sufficiently small, we have $\overline{B_\delta(z_k)}$, $k = 1, ..., N$, are contained in Ω and mutually disjoint. Let γ_k be the boundary of $B_\delta(z_k)$ with counterclockwise orientation. Then

$$\int_\gamma f(z)dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k).$$

$\square$

Example 17.4.

(i) If f is analytic at z_0, then $\mathrm{Res}(f; z_0) = 0$.

(ii) Suppose that f has a pole of order less than or equal to m at z_0 for some $m \in \mathbb{N}$. The function

$$g(z) = (z - z_0)^m f(z)$$

has a removable singularity at z_0. We can extend the domain of g to include z_0 and have the Taylor series about z_0,

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{g^{(n)}(z_0)}{n!},$$

in a neighborhood of z_0. And hence, the Laurent series of f about z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{k=-m}^{\infty} a_{k+m} (z - z_0)^k$$

near z_0. The residue of f at z_0 is

$$\mathrm{Res}(f; z_0) = a_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Example 17.5. *Let $f(z) = \dfrac{e^z - 1}{z^4}$. Now we compute* $\mathrm{Res}(f; 0)$. *Using the argument in Example 17.4,*

$$f(z) = \frac{g(z)}{z^4}, \quad \textit{where } g(z) = e^z - 1.$$

Thus,

$$\mathrm{Res}(f; 0) = \frac{g^{(3)}(0)}{3!} = \frac{1}{6}.$$

And by the residue theorem

$$\int_\gamma f(z)dz = 2\pi i \, \mathrm{Res}(f; 0) = \frac{\pi i}{3}$$

for any simple closed curve γ surrounding 0.

Example 17.6. *Let* $f(z) = \dfrac{1}{z(z-2)^5}$*. In order to evaluate the integral*

$$\int_\gamma f(z)dz,$$

where γ *is the unit circle centered at* 2 *with counterclockwise orientation, we first compute* $\mathrm{Res}(f;2)$*. Since*

$$f(z) = \frac{g(z)}{(z-2)^5}, \quad \text{where } g(z) = \frac{1}{z}.$$

Thus,

$$\mathrm{Res}(f;2) = \frac{g^{(4)}(2)}{4!} = \frac{1}{32}.$$

By the residue theorem

$$\int_\gamma f(z)dz = 2\pi i \,\mathrm{Res}(f;2) = \frac{\pi i}{16}.$$

Example 17.7. *Let* $f(z) = \dfrac{1}{z(z-2)^5}$*. In order to evaluate the integral*

$$\int_\gamma f(z)dz,$$

where γ *is the circle centered at* 2 *with radius* 5*, counterclockwise oriented, we need to compute* $\mathrm{Res}(f;2)$ *and* $\mathrm{Res}(f;0)$*. In Example 17.6, we showed*

$$\mathrm{Res}(f;2) = \frac{1}{32}.$$

To compute $\mathrm{Res}(f;0)$*, we have*

$$f(z) = \frac{h(z)}{z}, \quad \text{where } h(z) = \frac{1}{(z-2)^5}.$$

Thus,

$$\mathrm{Res}(f;0) = \frac{h(0)}{0!} = -\frac{1}{32}.$$

By the residue theorem

$$\int_\gamma f(z)dz = 2\pi i \left(\mathrm{Res}(f;0) + \mathrm{Res}(f;2)\right) = 0.$$

Example 17.8. *Let* $f(z) = \dfrac{1}{1-\cos z}$*. Notice that*

$$1 - \cos z = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} z^{2k}.$$

Hence, f has a pole of order 2 at 0. Let

$$f(z) = \frac{g(z)}{z^2}, \quad \text{where } g(z) = \frac{z^2}{1 - \cos z}.$$

By L'Hospital rule,

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{2z}{\sin z} = \lim_{z\to 0} \frac{2}{\cos z} = 2.$$

The residue of f at 0 can be computed by

$$\operatorname{Res}(f;0) = \lim_{z\to 0} \frac{\dfrac{z^2}{1-\cos z} - 2}{z - 0} = 0.$$

Remark 17.9 (L'Hospital rule)**.** *If f and g are analytic at z_0 and $f(z_0) = g(z_0) = 0$, then*

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}$$

provided the limit on the right-hand side exists.

Example 17.10. *Let $f(z) = \cot z = \dfrac{\cos z}{\sin z}$. Singularities of f occur at $z = n\pi$, $n \in \mathbb{Z}$. For each $z = n\pi$, $n \in \mathbb{Z}$, it is a simple pole, i.e., pole of order 1, of $\dfrac{1}{\sin z}$, and hence a simple pole of f. Let*

$$f(z) = \frac{g(z)}{z - n\pi}, \quad \text{where } g(z) = \frac{(z - n\pi)\cos z}{\sin z}.$$

Then

$$\operatorname{Res}(f; n\pi) = \lim_{z\to n\pi} \frac{(z-n\pi)\cos z}{\sin z} = 1.$$

Example 17.11. *Let $f(z) = \dfrac{z - \sinh z}{z^2 \sinh z}$. Singularities of f occur at $z = n\pi i$, $n \in \mathbb{Z}\backslash\{0\}$. At each $z = n\pi i$, $n \in \mathbb{Z}\backslash\{0\}$, since*

$$n\pi i - \sinh n\pi i = n\pi i \neq 0$$

and

$$\left(z^2 \sinh z\right)'\Big|_{z=n\pi i} = 2n\pi i \sinh n\pi i + (n\pi i)^2 \cosh n\pi i = (-1)^{n+1} n^2 \pi^2 \neq 0,$$

$z = n\pi i$ is a simple pole of f. Let

$$f(z) = \frac{g(z)}{z - n\pi i}, \quad \text{where } g(z) = \frac{(z - n\pi i)(z - \sinh z)}{z^2 \sinh z}.$$

Then

$$\operatorname{Res}(f; n\pi i) = \lim_{z\to n\pi i} g(z) = \frac{(-1)^{n+1} i}{n\pi}.$$

18 Improper Integrals

Definition 18.1. *For a continuous real-valued function f defined on $[0, \infty)$ or $\mathbb{R}$, the improper integral of f is defined by*

$$\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx$$

and

$$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx, \tag{18.1}$$

respectively, provided the limits on the right-hand sides of the equalities exist. There is another value assigned to the improper integral in (18.1), called the Cauchy principal value of the integral, and defined by

$$\text{P.V.} \int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx.$$

Example 18.2. *To evaluate the integral*

$$\int_0^\infty \frac{dx}{x^6+1},$$

firstly, we let γ_R, $R > 1$, be the closed curve consisting of C_R, the upper-half circle centered at the origin with radius R, and l_R, the line segment from $-R$ to R. And we assume that γ_R is counterclockwise oriented. In the region enclosed by γ_R, there are three zeros of $x^6 + 1$, that is, $c_1 = e^{i\pi/6}$, $c_2 = e^{i3\pi/6} = i$ and $c_3 = e^{i5\pi/6}$. By residue theorem,

$$\int_{\gamma_R} \frac{dz}{z^6+1} = 2\pi i \sum_{k=1}^3 \text{Res}\left(\frac{1}{z^6+1}; c_k\right).$$

For each $k = 1, 2, 3$, c_k is a simple pole of $\dfrac{1}{z^6+1}$, and we have

$$\text{Res}\left(\frac{1}{z^6+1}; c_k\right) = \lim_{z\to c_k} \frac{z-c_k}{z^6+1} = \frac{1}{6c_k^5} = -\frac{c_k}{6}.$$

Therefore,

$$\int_{\gamma_R} \frac{dz}{z^6+1} = \frac{2\pi}{3}.$$

Notice that

$$\int_{l_R} \frac{dz}{z^6+1} = \int_{-R}^R \frac{dx}{x^6+1}.$$

And we have

$$\left|\int_{C_R} \frac{dz}{z^6+1}\right| \le \frac{\pi R}{R^6-1} \longrightarrow 0 \quad \text{as } R \to \infty.$$

By passing to the limit $R \to \infty$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

Since $\dfrac{1}{x^6+1}$ is even, we have

$$\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

Example 18.3. *Now, we want to evaluate integrals of the form*

$$\int_{-\infty}^{\infty} f(x) \sin ax dx \quad or \quad \int_{-\infty}^{\infty} f(x) \cos ax dx, \quad a > 0.$$

In view of Euler's formula, it is equivalent to consider

$$\int_{-\infty}^{\infty} f(x) e^{iax} dx.$$

These integrals occur in the theory of Fourier analysis. Let γ_R, C_R and l_R be defined as in Example 18.2. If $z_1, ..., z_N$ are all the singularities of $f(z)e^{iaz}$ in the region enclosed by γ_R for R large. By residue theorem,

$$\int_{\gamma_R} f(z) e^{iaz} dz = 2\pi i \sum_{k=1}^{N} \text{Res}\left(f(z)e^{iaz}; z_k\right).$$

Therefore, we have

$$\int_{-R}^{R} f(x) e^{iax} dx = \int_{l_R} f(z) e^{iaz} dz = 2\pi i \sum_{k=1}^{N} \text{Res}\left(f(z)e^{iaz}; z_k\right) - \int_{C_R} f(z) e^{iaz} dz. \tag{18.2}$$

Example 18.4. *To evaluate the integral*

$$\int_0^{\infty} \frac{\cos 2x}{(x^2+4)^2} dx,$$

we follow the argument in Example 18.3 with

$$f(z) = \frac{1}{(z^2+4)^2} \quad and \quad a = 2.$$

Notice that $2i$ is the only singularity of $\dfrac{e^{i2z}}{(z^2+4)^2}$ in the region enclosed by γ_R for R large. Then (18.2) becomes

$$\int_{-R}^{R} \frac{e^{i2x}}{(x^2+4)^2} dx = 2\pi i \,\text{Res}\left(\frac{e^{i2z}}{(z^2+4)^2}; 2i\right) - \int_{C_R} \frac{e^{i2z}}{(z^2+4)^2} dz.$$

On one hand, $2i$ is a pole of order 2 of $\dfrac{e^{i2z}}{(z^2+4)^2}$*. By letting*

$$\frac{e^{i2z}}{(z^2+4)^2} = \frac{g(z)}{(z-2i)^2}, \quad \textit{where } g(z) = \frac{e^{i2z}}{(z+2i)^2},$$

we have

$$\operatorname{Res}\left(\frac{e^{i2z}}{(z^2+4)^2}; 2i\right) = g'(2i) = \frac{5}{32e^4 i}.$$

On the other hand,

$$\left|\int_{C_R} \frac{e^{i2z}}{(z^2+4)^2} dz\right| \le \frac{\pi R}{(R^2-4)^2} \longrightarrow 0 \quad \textit{as } R \to \infty.$$

Therefore, by passing to the limit $R \to \infty$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i2x}}{(x^2+4)^2} dx = 2\pi i \cdot \frac{5}{32e^4 i} = \frac{5\pi}{16e^4}.$$

Taking the real parts on both sides above yields

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5\pi}{16e^4}.$$

Since $\dfrac{\cos 2x}{(x^2+4)^2}$ *is even,*

$$\int_0^{\infty} \frac{\cos 2x}{(x^2+4)^2} dx = \frac{5\pi}{32e^4}.$$

Lemma 18.5 (Jordan's lemma)**.** *Let C_R be defined as in Example 18.2. Suppose that*

(i) f is analytic on $\{z \in \mathbb{C} : \operatorname{Im} z \ge 0, |z| \ge R_0\}$ for some $R_0 > 0$;

(ii) For each $R > R_0$, there is a positive constant M_R such that

$$\max_{C_R} |f| \le M_R,$$

and

$$\lim_{R\to\infty} M_R = 0.$$

Then, for every $a > 0$,

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iaz} dz = 0.$$

Proof. For $a > 0$, $R > R_0$,

$$\begin{aligned}\int_{C_R} f(z)e^{iaz}dz &= \int_0^\pi f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} \cdot iRe^{i\theta} d\theta \\ &= iR \int_0^\pi f\left(Re^{i\theta}\right) e^{-aR\sin\theta} e^{iaR\cos\theta} e^{i\theta} d\theta.\end{aligned}$$

Thus,

$$\left| \int_{C_R} f(z)e^{iaz}dz \right| \le RM_R \int_0^\pi e^{-aR\sin\theta} d\theta.$$

Notice that

$$\int_0^\pi e^{-aR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-aR\sin\theta} d\theta.$$

By using the fact that $\sin\theta \ge \dfrac{2\theta}{\pi}$ for $\theta \in \left[0, \dfrac{\pi}{2}\right]$,

$$\int_0^{\pi/2} e^{-aR\sin\theta} d\theta \le \int_0^{\pi/2} e^{-2aR\theta/\pi} d\theta = \frac{\pi}{2aR}\left(1 - e^{-aR}\right) \le \frac{\pi}{2aR}.$$

Therefore,

$$\left| \int_{C_R} f(z)e^{iaz}dz \right| \le RM_R \cdot \frac{\pi}{aR} \longrightarrow 0 \quad \text{as } R \to \infty.$$

□

Example 18.6. *To evaluate the integral*

$$\int_0^\infty \frac{x\sin 2x}{x^2+3} dx,$$

we follow the argument in Example 18.3 with

$$f(z) = \frac{z}{z^2+3} \quad \textit{and} \quad a = 2.$$

Notice that $\sqrt{3}i$ *is the only singularity of* $\dfrac{ze^{i2z}}{z^2+3}$ *enclosed by* γ_R *for* R *large. Then we have*

$$\int_{-R}^R \frac{xe^{i2x}}{x^2+3} dx = 2\pi i \operatorname{Res}\left(\frac{ze^{i2z}}{z^2+3}; \sqrt{3}i\right) - \int_{C_R} \frac{ze^{i2z}}{z^2+3} dz.$$

Let

$$\frac{ze^{i2z}}{z^2+3} = \frac{g(z)}{z-\sqrt{3}i}, \quad \textit{where } g(z) = \frac{ze^{i2z}}{z+\sqrt{3}i},$$

then

$$\operatorname{Res}\left(\frac{ze^{i2z}}{z^2+3}; \sqrt{3}i\right) = g\left(\sqrt{3}i\right) = \frac{1}{2e^{2\sqrt{3}}}.$$

Moreover,

$$\max_{C_R} |f| \leq \frac{R}{R^2-3} \longrightarrow 0 \quad \textit{as } R \to \infty.$$

By Jordan's lemma,

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{i2z}}{z^2+3} dz = 0.$$

Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{xe^{i2x}}{x^2+3} dx = 2\pi i \cdot \frac{1}{2e^{2\sqrt{3}}} = \frac{\pi i}{e^{2\sqrt{3}}}.$$

Taking the imaginary parts on both sides above yields

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3} dx = \frac{\pi}{e^{2\sqrt{3}}}.$$

Since $\dfrac{x \sin 2x}{x^2+3}$ *is even,*

$$\int_0^{\infty} \frac{x \sin 2x}{x^2+3} dx = \frac{\pi}{2e^{2\sqrt{3}}}.$$

Example 18.7. *To evaluate the Dirichlet integral*

$$\int_0^{\infty} \frac{\sin x}{x} dx,$$

we let $\gamma_{\rho,R}$*,* $0 < \rho < R$*, be the closed curve consisting of* C_R*,* $l_{-R,-\rho}$*,* C_ρ *and* $l_{\rho,R}$ *with counter-clockwise orientation, where* C_R *is the upper-half circle centered at origin with radius* R *from* R *to* $-R$*,* C_ρ *is the upper-half circle centered at origin with radius* ρ *from* $-\rho$ *to* ρ*,* $l_{-R,-\rho}$ *is the line segment from* $-R$ *to* $-\rho$*, and* $l_{\rho,R}$ *is the line segment from* ρ *to* R*. By Cauchy-Goursat theorem,*

$$\int_{\gamma_R} \frac{e^{iz}}{z} dz = 0.$$

That is,

$$\int_{C_R} \frac{e^{iz}}{z} dz + \int_{l_{-R,-\rho}} \frac{e^{iz}}{z} dz + \int_{C_\rho} \frac{e^{iz}}{z} dz + \int_{l_{\rho,R}} \frac{e^{iz}}{z} dz = 0.$$

The last equality can be further rewritten as

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz. \tag{18.3}$$

For the left-hand side of (18.3), we have

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx = 2i \int_{\rho}^{R} \frac{\sin x}{x} dx. \tag{18.4}$$

For the right-hand side of (18.3), first, $-C_\rho$ can be parametrized by $\rho e^{i\theta}$, $\theta \in [0, \pi]$, and hence

$$\int_{C_\rho} \frac{e^{iz}}{z} dz = -\int_0^\pi \frac{e^{i\rho e^{i\theta}}}{\rho e^{i\theta}} \cdot i\rho e^{i\theta} d\theta = -i \int_0^\pi e^{i\rho e^{i\theta}} d\theta.$$

By using the fact that

$$|e^z - 1| \leq C|z| \quad \textit{for all } |z| \leq 1$$

for some positive constant C, we have, for $\rho \leq 1$,

$$\left| \int_0^\pi \left(e^{i\rho e^{i\theta}} - 1 \right) d\theta \right| \leq \int_0^\pi \left| e^{i\rho e^{i\theta}} - 1 \right| d\theta \leq \int_0^\pi C\rho d\theta = C\pi\rho.$$

Thus,

$$\lim_{\rho \to 0} \int_0^\pi e^{i\rho e^{i\theta}} d\theta = \pi,$$

which gives

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -\pi i.$$

Second, by Jordan's lemma,

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0.$$

By (18.4) and the last two equalities, (18.3) implies

$$2i \int_0^\infty \frac{\sin x}{x} dx = \pi i.$$

That is,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example 18.8. *For $-1 < a < 3$, to evaluate the integral*

$$\int_0^\infty \frac{x^a}{(x^2+1)^2} dx,$$

we let

$$f(z) = \frac{z^a}{(z^2+1)^2},$$

where $z^a = e^{a \log z}$ is defined by using the branch of logarithm

$$\log z = \ln |z| + i\theta, \quad \theta \in \arg z, \quad \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right).$$

In addition, we let $\gamma_{\rho,R}$, C_R, C_ρ, $l_{-R,-\rho}$ and $l_{\rho,R}$ be defined as in Example 18.7 with $0 < \rho < 1 < R$. Since i is the only singularity of f in the region enclosed by $\gamma_{\rho,R}$, by residue theorem,

$$\int_{\gamma_{\rho,R}} f(z)dz = 2\pi i \operatorname{Res}\left(f(z); i\right). \tag{18.5}$$

By letting

$$f(z) = \frac{g(z)}{(z-i)^2}, \quad \textit{where } g(z) = \frac{z^a}{(z+i)^2},$$

we have

$$\operatorname{Res}\left(f(z); i\right) = g'(i) = \left.\frac{(a-2)z^a + iaz^{a-1}}{(z+i)^3}\right|_{z=i} = \frac{1-a}{4}\, e^{(a-1)\log i} = ie^{ia\pi/2}\left(\frac{a-1}{4}\right).$$

We can divide the integral along $\gamma_{\rho,R}$ into four parts.

$$\int_{\gamma_{\rho,R}} f(z)dz = \int_{C_R} \frac{z^a}{(z^2+1)^2}dz + \int_{l_{-R,-\rho}} \frac{z^a}{(z^2+1)^2}dz + \int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz + \int_{l_{\rho,R}} \frac{z^a}{(z^2+1)^2}dz.$$

First,

$$\int_{l_{\rho,R}} \frac{z^a}{(z^2+1)^2}dz = \int_\rho^R \frac{x^a}{(x^2+1)^2}dx.$$

Second,

$$\int_{l_{-R,-\rho}} \frac{z^a}{(z^2+1)^2}dz = \int_{-R}^{-\rho} \frac{e^{a(\ln|x|+i\pi)}}{(x^2+1)^2}dx = e^{ia\pi}\int_\rho^R \frac{x^a}{(x^2+1)^2}dx.$$

Third,

$$\begin{aligned}
\int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz &= -\int_{-C_\rho} \frac{z^a}{(z^2+1)^2}dz \\
&= -\int_0^\pi \frac{e^{a\log\left(\rho e^{i\theta}\right)}}{\left(\rho^2 e^{2i\theta}+1\right)^2}\cdot i\rho e^{i\theta}d\theta \\
&= -i\rho\int_0^\pi \frac{e^{a(\ln\rho+i\theta)}}{\left(\rho^2 e^{2i\theta}+1\right)^2}\cdot e^{i\theta}d\theta \\
&= -i\rho^{a+1}\int_0^\pi \frac{e^{i(a+1)\theta}}{\left(\rho^2 e^{2i\theta}+1\right)^2}d\theta.
\end{aligned}$$

Thus,

$$\left|\int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz\right| \le \frac{\pi\rho^{a+1}}{\left(1-\rho^2\right)^2},$$

which implies

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^a}{(z^2+1)^2}dz = 0.$$

Finally,

$$\begin{aligned}\int_{C_R} \frac{z^a}{(z^2+1)^2} dz &= \int_0^\pi \frac{e^{a \log\left(Re^{i\theta}\right)}}{\left(R^2 e^{2i\theta}+1\right)^2} \cdot iRe^{i\theta} d\theta \\ &= iR \int_0^\pi \frac{e^{a(\ln R + i\theta)}}{\left(R^2 e^{2i\theta}+1\right)^2} \cdot e^{i\theta} d\theta \\ &= iR^{a+1} \int_0^\pi \frac{e^{i(a+1)\theta}}{\left(R^2 e^{2i\theta}+1\right)^2} d\theta.\end{aligned}$$

Thus,

$$\left|\int_{C_R} \frac{z^a}{(z^2+1)^2} dz\right| \le \frac{\pi R^{a+1}}{\left(R^2-1\right)^2},$$

which implies

$$\lim_{R\to\infty} \int_{C_R} \frac{z^a}{(z^2+1)^2} dz = 0.$$

By passing to the limit $\rho \to 0$ *and* $R \to \infty$, *(18.5) becomes*

$$\left(1+e^{ia\pi}\right) \int_0^\infty \frac{x^a}{(x^2+1)^2} dx = 2\pi i \cdot i e^{ia\pi/2} \left(\frac{a-1}{4}\right) = \frac{\pi(1-a)e^{ia\pi/2}}{2}.$$

Therefore, if $a \neq 1$,

$$\int_0^\infty \frac{x^a}{(x^2+1)^2} dx = \frac{\pi(1-a)e^{ia\pi/2}}{2\left(1+e^{ia\pi}\right)} = \frac{\pi(1-a)}{2\left(e^{-ia\pi/2}+e^{ia\pi/2}\right)} = \frac{\pi(1-a)}{4\cos(a\pi/2)}.$$

For $a = 1$, *by change of variables* $y = x^2+1$,

$$\begin{aligned}\int_0^\infty \frac{x^a}{(x^2+1)^2} dx &= \int_0^\infty \frac{x}{(x^2+1)^2} dx \\ &= \lim_{R\to\infty} \int_0^R \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \lim_{R\to\infty} \int_1^{R^2+1} \frac{dy}{y^2} = \frac{1}{2} \lim_{R\to\infty} \left(1 - \frac{1}{R^2+1}\right) = \frac{1}{2}.\end{aligned}$$

Example 18.9. *For* $0 < a < 1$, *to evaluate the integral*

$$\int_0^\infty \frac{x^{-a}}{x+1} dx,$$

we let

$$f(z) = \frac{z^{-a}}{z+1},$$

where $z^{-a} = e^{-a \log z}$ *is defined by using the branch of logarithm*

$$\log z = \ln|z| + i\theta, \quad \theta \in \arg z, \quad \theta \in (0, 2\pi).$$

Moreover, for $0 < \rho < 1 < R$ *and* $0 < \alpha < \dfrac{\pi}{2}$, *we define* $\gamma_{\rho,R,\alpha}$ *to be the simple closed curve consisting of*

(i) C_R, *the arc on the circle* $\partial B_R(0)$ *from* $Re^{i\alpha}$ *to* $Re^{-i\alpha}$ *counterclockwise,*

(ii) l_-, *the line segment from* $Re^{-i\alpha}$ *to* $\rho e^{-i\alpha}$,

(iii) C_ρ, *the arc on the circle* $\partial B_\rho(0)$ *from* $\rho e^{-i\alpha}$ *to* $\rho e^{i\alpha}$ *clockwise, and*

(iv) l_+, *the line segment from* $\rho e^{i\alpha}$ *to* $Re^{i\alpha}$.

Since -1 *is the only singularity of* f *in the region enclosed by* $\gamma_{\rho,R,\alpha}$, *by residue theorem,*

$$\int_{\gamma_{\rho,R,\alpha}} f(z)dz = 2\pi i \operatorname{Res}(f(z); -1). \tag{18.6}$$

Let

$$f(z) = \frac{g(z)}{z+1}, \quad \textit{where } g(z) = z^{-a},$$

then we have

$$\operatorname{Res}(f(z); -1) = g(-1) = (-1)^{-a} = e^{-ia\pi}.$$

Divide the integral along $\gamma_{\rho,R,\alpha}$ *into four parts*

$$\int_{\gamma_{\rho,R,\alpha}} f(z)dz = \int_{C_R} \frac{z^{-a}}{z+1}dz + \int_{l_-} \frac{z^{-a}}{z+1}dz + \int_{C_\rho} \frac{z^{-a}}{z+1}dz + \int_{l_+} \frac{z^{-a}}{z+1}dz.$$

First,

$$\begin{aligned}
\int_{C_R} \frac{z^{-a}}{z+1}dz &= \int_\alpha^{2\pi-\alpha} \frac{\left(Re^{i\theta}\right)^{-a}}{Re^{i\theta}+1} \cdot iRe^{i\theta}d\theta \\
&= iR\int_\alpha^{2\pi-\alpha} \frac{e^{-a(\ln R+i\theta)}}{Re^{i\theta}+1}e^{i\theta}d\theta \\
&= iR^{1-a}\int_\alpha^{2\pi-\alpha} \frac{e^{i(1-a)\theta}}{Re^{i\theta}+1}d\theta
\end{aligned}$$

Second,

$$\begin{aligned}
\int_{l_-} \frac{z^{-a}}{z+1}dz &= \int_\rho^R \frac{((R+\rho-r)e^{-i\alpha})^{-a}}{(R+\rho-r)e^{-i\alpha}+1} \cdot (-e^{-i\alpha})dr \\
&= -e^{-i\alpha}\int_\rho^R \frac{e^{-a(\ln(R+\rho-r)+i(2\pi-\alpha))}}{(R+\rho-r)e^{-i\alpha}+1}dr \\
&= -e^{-i2a\pi-i(1-a)\alpha}\int_\rho^R \frac{(R+\rho-r)^{-a}}{(R+\rho-r)e^{-i\alpha}+1}dr \\
&= -e^{-i2a\pi-i(1-a)\alpha}\int_\rho^R \frac{r^{-a}}{re^{-i\alpha}+1}dr.
\end{aligned}$$

Third,

$$\begin{aligned}\int_{C_\rho} \frac{z^{-a}}{z+1}dz &= \int_\alpha^{2\pi-\alpha} \frac{\left(\rho e^{i(2\pi-\theta)}\right)^{-a}}{\rho e^{i(2\pi-\theta)}+1} \cdot \left(-i\rho e^{i(2\pi-\theta)}\right) d\theta \\ &= -i\rho \int_\alpha^{2\pi-\alpha} \frac{e^{-a(\ln\rho+i(2\pi-\theta))}}{\rho e^{i(2\pi-\theta)}+1} e^{-i\theta} d\theta \\ &= -i\rho^{1-a} \int_\alpha^{2\pi-\alpha} \frac{e^{i(a-1)\theta-i2a\pi}}{\rho e^{i(2\pi-\theta)}+1} d\theta.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{l_+} \frac{z^a}{(z^2+1)^2}dz &= \int_\rho^R \frac{(re^{i\alpha})^{-a}}{re^{i\alpha}+1} \cdot e^{i\alpha} dr \\ &= e^{i\alpha} \int_\rho^R \frac{e^{-a(\ln r+i\alpha)}}{re^{i\alpha}+1} dr \\ &= e^{i(1-a)\alpha} \int_\rho^R \frac{r^{-a}}{re^{i\alpha}+1} dr.\end{aligned}$$

By passing to the limit $\alpha \to 0$, (18.6) implies

$$\left(1-e^{-i2a\pi}\right) \int_\rho^R \frac{r^{-a}}{r+1} dr + iR^{1-a} \int_0^{2\pi} \frac{e^{i(1-a)\theta}}{Re^{i\theta}+1} d\theta - i\rho^{1-a} \int_0^{2\pi} \frac{e^{i(a-1)\theta-i2a\pi}}{\rho e^{i(2\pi-\theta)}+1} d\theta = 2\pi i e^{-ia\pi}.$$

Since

$$\left| iR^{1-a} \int_0^{2\pi} \frac{e^{i(1-a)\theta}}{Re^{i\theta}+1} d\theta \right| \le R^{1-a} \cdot \frac{2\pi}{R-1} \longrightarrow 0 \quad \text{as } R \to \infty$$

and

$$\left| i\rho^{1-a} \int_0^{2\pi} \frac{e^{i(a-1)\theta-i2a\pi}}{\rho e^{i(2\pi-\theta)}+1} d\theta \right| \le \rho^{1-a} \cdot \frac{2\pi}{1-\rho} \longrightarrow 0 \quad \text{as } \rho \to 0,$$

by passing to the limits $\rho \to 0$ and $R \to \infty$, we have

$$\left(1-e^{-i2a\pi}\right) \int_0^\infty \frac{r^{-a}}{r+1} dr = 2\pi i e^{-ia\pi}.$$

That is,

$$\int_0^\infty \frac{r^{-a}}{r+1} dr = \frac{2\pi i}{e^{ia\pi}-e^{-ia\pi}} = \frac{\pi}{\sin a\pi}.$$

19 Definite Integrals

Example 19.1. *Residue theorem can be used to evaluate definite integrals of the form*

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta.$$

Let $z(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$, *then* $\dfrac{1}{z(\theta)} = e^{-i\theta}$. *Thus,*

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z(\theta) - \dfrac{1}{z(\theta)}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z(\theta) + \dfrac{1}{z(\theta)}}{2},$$

and hence,

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_0^{2\pi} F\left(\frac{z(\theta) - \dfrac{1}{z(\theta)}}{2i}, \frac{z(\theta) + \dfrac{1}{z(\theta)}}{2} \right) d\theta.$$

Since $z'(\theta) = ie^{i\theta} = iz(\theta)$, *by letting* γ *be the unit circle centered at the origin with counterclockwise orientation, we further have*

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_\gamma F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2} \right) \frac{1}{iz} dz.$$

Example 19.2. *To evaluate*

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, \qquad -1 < a < 1,$$

we have, for $a \neq 0$,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \int_\gamma \frac{1}{1 + a(z - z^{-1})/2i} \cdot \frac{1}{iz} dz = \int_\gamma \frac{2/a}{z^2 + 2iz/a - 1} dz,$$

where γ *is the unit circle centered at the origin with counterclockwise orientation. The polynomial* $z^2 + 2iz/a + 1$ *has pure imaginary roots*

$$z_1 = \left(\frac{-1 + \sqrt{1 - a^2}}{a} \right) i \quad \text{and} \quad z_2 = \left(\frac{-1 - \sqrt{1 - a^2}}{a} \right) i.$$

Thus,

$$f(z) := \frac{2/a}{z^2 + 2iz/a - 1} = \frac{2/a}{(z - z_1)(z - z_2)}.$$

Since $|a| < 1$,

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1.$$

Moreover, since $|z_1||z_2| = 1$, *we have*

$$|z_1| < 1.$$

Hence, the only singularity of f *in the region enclosed by* γ *is* z_1. *Let*

$$f(z) = \frac{g(z)}{z - z_1}, \quad \text{where } g(z) = \frac{2/a}{z - z_2}.$$

Then

$$\operatorname{Res}(f; z_1) = g(z_1) = \frac{1}{i\sqrt{1-a^2}}.$$

By residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \int_\gamma \frac{2/a}{z^2 + 2iz/a - 1} dz = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

As for the case $a = 0$,

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \int_0^{2\pi} 1 d\theta = 2\pi.$$

We conclude that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for } -1 < a < 1.$$

Example 19.3. *To evaluate*

$$\int_0^{\pi} \frac{\cos 2\theta d\theta}{1 - 2a\cos\theta + a^2}, \quad -1 < a < 1,$$

we have

$$\begin{aligned}
\int_0^{\pi} \frac{\cos 2\theta d\theta}{1 - 2a\cos\theta + a^2} &= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1 - 2a\cos\theta + a^2} \\
&= \frac{1}{2}\int_0^{2\pi} \frac{\left(2\cos^2\theta - 1\right) d\theta}{1 - 2a\cos\theta + a^2} \\
&= \frac{1}{2}\int_\gamma \frac{\left(z + z^{-1}\right)^2/2 - 1}{1 - a\left(z + z^{-1}\right) + a^2} \cdot \frac{1}{iz} dz \\
&= \frac{i}{4}\int_\gamma \frac{z^4 + 1}{(z-a)(az-1)z^2} dz,
\end{aligned}$$

where γ is the unit circle centered at the origin with counterclockwise orientation. If $a \neq 0$, let

$$f(z) = \frac{z^4 + 1}{(z-a)(az-1)z^2},$$

then singularities of f that lie in the region enclosed by γ are 0 and a. For the residue of f at 0, we let

$$f(z) = \frac{g(z)}{z^2}, \quad \text{where } g(z) = \frac{z^4 + 1}{(z-a)(az-1)},$$

then

$$\operatorname{Res}(f; 0) = g'(0) = \frac{a^2 + 1}{a^2}.$$

For the residue of f at a, we let

$$f(z) = \frac{h(z)}{z-a}, \quad \text{where } h(z) = \frac{z^4+1}{(az-1)z^2},$$

then

$$\operatorname{Res}(f;a) = h(a) = \frac{a^4+1}{(a^2-1)\,a^2}.$$

By residue theorem,

$$\begin{aligned}\int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} &= \frac{i}{4}\int_\gamma \frac{z^4+1}{(z-a)(az-1)z^2}dz\\ &= -\frac{\pi}{2}\left(\frac{a^2+1}{a^2} + \frac{a^4+1}{(a^2-1)\,a^2}\right) = \frac{\pi a^2}{1-a^2}.\end{aligned}$$

As for the case $a=0$,

$$\int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \int_0^\pi \cos 2\theta d\theta = 0.$$

That is,

$$\int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} = \frac{\pi a^2}{1-a^2} \quad \text{for } -1<a<1.$$

20 Argument Principle

Definition 20.1. *A function f is meromorphic in an open connected set Ω if f is analytic throughout Ω except for poles.*

Definition 20.2. *Given a curve γ from z_1 to z_2 parametrized by $z(t), t \in [a,b]$, if f is analytic and has no zero on γ, we define $\Delta_\gamma \arg f(z)$ to be the continuous change in $\arg f(z)$ along γ from z_1 to z_2. That is,*

$$\Delta_\gamma \arg f(z) = \theta(b) - \theta(a),$$

provided that

$$f(z(t)) = \rho(t)e^{i\theta(t)}, \quad t \in [a,b], \tag{20.1}$$

where ρ and θ are continuous on $[a,b]$.

Remark 20.3.

(i) $\rho(t) = |f(z(t))|$ can be uniquely determined. As for $\theta(t)$, we only have $\theta(t) \in \arg f(z(t))$ for each $t \in [a,b]$.

(ii) If θ and $\tilde{\theta}$ are both continuous and satisfy (20.1), then we can show that $\theta(t) = \tilde{\theta(t)} + 2k\pi$ for some $k \in \mathbb{Z}$. Thus, the definition of $\Delta_\gamma \arg f(z)$ is independent of the choice of θ.

(iii) If there is $\alpha \in \mathbb{R}$ such that $f(\gamma) \cap \{re^{i\alpha} : r \geq 0\} = \phi$, then we can use

$$\theta(t) = \theta_\alpha(t),$$

where

$$\theta_\alpha(t) \in \arg f(z(t)) \cap (\alpha, \alpha + 2\pi),$$

in (20.1).

(iv) Suppose that no such α in (iii) exists. For each $t \in [a,b]$, we can find an open interval I_t containing t such that the assumption in (iii) holds if we replace γ by $f(z(I_t \cap [a,b]))$. Since $[a,b]$ is compact, we can find finitely many t_1, t_2, ..., $t_N \in [a,b]$, such that $[a,b] \subset \cup_{j=1}^N I_j$, where $I_j = I_{t_j}$. Without loss of generality, we assume that

$$a \in I_1, \quad b \in I_N, \quad \text{and there is} \quad s_j \in I_j \cap I_{j+1} \neq \phi \text{ for all } j = 1, ..., N-1.$$

On each $I_j \cap [a,b]$, $j = 1, ..., N$, there is a continuous function θ_j such that

$$\theta_j(t) \in \arg f(z(t)) \quad \text{for all } t \in I_j \cap [a,b].$$

We let $\Theta_1 = \theta_1$ on $I_1 \cap [a,b]$. Inductively, for Θ_j, $j = 1, ..., N-1$, already defined, we choose $k_{j+1} \in \mathbb{Z}$ such that $\theta_{j+1}(s_j) + 2k_{j+1}\pi = \Theta_j(s_j)$ and let

$$\Theta_{j+1}(t) = \theta_{j+1}(t) + 2k_{j+1}\pi \quad \text{for all } t \in I_{j+1} \cap [a,b].$$

Therefore, we can let

$$\theta(t) = \Theta_j(t) \quad \text{if } t \in I_j \cap [a,b],$$

which is a continuous function satisfying (20.1).

Remark 20.4. *Suppose that γ is a simple closed curve with counterclockwise orientation. f is meromorphic and has no zero and no pole on γ. Then $\Delta_\gamma \arg f(z)$ is independent of the choice of the starting point of γ. Moreover, it is an integral multiple of 2π. The integer*

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z)$$

represents the number of the times that the image of γ under f winds around the origin. It is positive if the image winds around the origin in counterclockwise direction, and it is negative if the image winds around the origin in clockwise direction. Moreover, if the image $f(\gamma)$ does not enclose the origin, then

$$\Delta_\gamma \arg f(z) = 0.$$

Theorem 20.5 (argument principle)**.** *Suppose that f is meromorphic on an open connected set Ω. Let γ be a simple closed curve with counterclockwise orientation in Ω. If f has no pole and no zero on γ, then*

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = Z - P,$$

where Z is the number of zeros of f inside γ, and P is the number of poles of f inside γ, both counting multiplicities.

Proof. Let γ be parametrized by $z(t)$, $t \in [a, b]$. We have

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(z(t))z'(t)}{f(z(t))} dt.$$

Notice that

$$f(z(t)) = \rho(t)e^{i\theta(t)}, \quad t \in [a, b],$$

for ρ and θ smooth. We have

$$f'(z(t))z'(t) = \frac{d}{dt} f(z(t)) = \frac{d}{dt}\rho(t)e^{i\theta(t)} = \rho'(t)e^{i\theta(t)} + i\rho(t)e^{i\theta(t)}\theta'(t).$$

Therefore,

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)} dz &= \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i \int_a^b \theta'(t) dt \\ &= \ln \rho(b) - \ln \rho(a) + i\left(\theta(b) - \theta(a)\right) = i\Delta_\gamma \arg f(z). \end{aligned} \tag{20.2}$$

On the other hand, in the region enclosed by γ, suppose f has zeros Z_j of order m_j, $j = 1, ..., M$, and poles P_k of order n_k, $k = 1, ..., N$. Then near each Z_j, $j = 1, ..., M$,

$$f(z) = (z - Z_j)^{m_j} g_j(z)$$

for some g_j analytic at Z_j with $g_j(Z_j) \neq 0$. Hence, near Z_j,

$$\frac{f'(z)}{f(z)} = \frac{m_j(z - Z_j)^{m_j - 1} g_j(z) + (z - Z_j)^{m_j} g_j'(z)}{(z - Z_j)^{m_j} g_j(z)} = \frac{m_j}{z - Z_j} + \frac{g_j'(z)}{g_j(z)}.$$

That is, f'/f has a simple pole with residue m_j at Z_j, $j = 1, ..., M$. Near each P_k, $k = 1, ..., N$,

$$f(z) = \frac{h_k(z)}{(z - P_k)^{n_k}}$$

for some h_k analytic at P_k with $h_k(P_k) \neq 0$. Hence, near P_k,

$$\frac{f'(z)}{f(z)} = \frac{-n_k(z - P_k)^{-n_k - 1} h_k(z) + (z - P_k)^{-n_k} h_k'(z)}{(z - P_k)^{-n_k} h_k(z)} = -\frac{n_k}{z - P_k} + \frac{h_k'(z)}{h_k(z)}.$$

That is, f'/f has a simple pole with residue $-n_k$ at P_k, $k = 1, ..., N$. By residue theorem,

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \left(\sum_{j=1}^M m_j - \sum_{k=1}^N n_k \right) = 2\pi i \left(Z - P\right). \tag{20.3}$$

Combining (20.2) and (20.3), we complete the proof. □

Example 20.6. *Let*

$$f(z) = \frac{z^3 + 2}{z} = z^2 + \frac{2}{z}.$$

f has a simple pole at 0, and all zeros of f are outside the unit disc centered at the origin. Let γ be the unit circle centered at the origin with counterclockwise orientation. Then the argument principle tells us that

$$\frac{1}{2\pi} \Delta_\gamma \arg f(z) = 0 - 1 = -1.$$

That is, the image of γ under f winds around the origin once in the clockwise direction.

Example 20.7. *In this example, we determine the number of roots of the equation* $P(z) = z^4 + 8z^3 + 3z^2 + 2z + 2 = 0$ *on the right-half plane* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$*. For* $R > 0$*, let* γ_R *be the closed curve consisting the line segment* l_R *from* iR *to* $-iR$ *and the arc* C_R *on* $\partial B_R(0)$ *from* $-iR$ *to* iR *counterclockwise. For any pure imaginary number* ci*,* $c \in \mathbb{R}$*,*

$$P(ci) = (ci)^4 + 8(ci)^3 + 3(ci)^2 + 2(ci) + 2 = c^4 - 3c^2 + 2 + \left(-8c^3 + 2c\right) i.$$

Since $c^4 - 3c^2 + 2$ *and* $-8c^3 + 2c$ *have no common zero, the equation* $P(z) = 0$ *has no root on the imaginary axis. Also, by the fundamental theorem of algebra, the equation has exact four roots. Thus,* $P(z)$ *has no zero on* C_R *provided* R *sufficiently large. Since* P *is analytic, by the argument principle,*

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{P'(z)}{P(z)} dz$$

equals to the number of zeros of $P(z)$ *in the right-half plane provided* R *large. First, as in (20.2)*

$$\int_{l_R} \frac{P'(z)}{P(z)} dz = i\Delta_{l_R} \arg f(z).$$

Notice that $\operatorname{Im} P(ci) = 0$ *at* $c = -1/2$*,* 0*, or* $1/2$*. At these points,* $P(z)$ *takes values* $21/16$*,* 2*, and* $21/16$*, respectively. Thus, the image of* l_R *under* P *does not intersect the half line* $\{x \in \mathbb{R} : x \le 0\}$*. As a consequence, for* R *sufficiently large,*

$$\begin{aligned}
\Delta_{l_R} \arg f(z) &= \operatorname{Arg}\left(R^4 - 3R^2 + 2 + \left(8R^3 - 2R\right) i\right) - \operatorname{Arg}\left(R^4 - 3R^2 + 2 - \left(8R^3 - 2R\right) i\right) \\
&= \arctan\left(\frac{8R^3 - 2R}{R^4 - 3R^2 + 2}\right) - \arctan\left(\frac{-8R^3 + 2R}{R^4 - 3R^2 + 2}\right) \\
&\longrightarrow 0 \quad \text{as } R \to 0.
\end{aligned}$$

That is,

$$\lim_{R\to\infty} \int_{l_R} \frac{P'(z)}{P(z)} dz = 0. \tag{20.4}$$

Second,

$$\int_{C_R} \frac{P'(z)}{P(z)} dz = \int_{-\pi/2}^{\pi/2} \frac{4R^3e^{i3\theta} + 24R^2e^{i2\theta} + 6Re^{i\theta} + 2}{R^4e^{i4\theta} + 8R^3e^{i3\theta} + 3R^2e^{i2\theta} + 2Re^{i\theta} + 2} \cdot iRe^{i\theta} d\theta.$$

Since the integrand in the last integral

$$\frac{4R^3e^{i3\theta} + 24R^2e^{i2\theta} + 6Re^{i\theta} + 2}{R^4e^{i4\theta} + 8R^3e^{i3\theta} + 3R^2e^{i2\theta} + 2Re^{i\theta} + 2} \cdot iRe^{i\theta} \longrightarrow 4i \quad \text{as } R \to \infty,$$

uniformly in $\theta \in [-\pi/2, \pi/2]$*. We have*

$$\lim_{R\to\infty} \int_{C_R} \frac{P'(z)}{P(z)} dz = 4\pi i. \tag{20.5}$$

Combining (20.4) and (20.5), we obtain

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{P'(z)}{P(z)} dz = 4\pi i.$$

Therefore, there are two roots of $P(z) = 0$ *on the right-half plane.*

21 Rouché's Theorem

Theorem 21.1 (Rouché's theorem)**.** *Let γ be a simple closed curve. Suppose that f and g are analytic on the closure of the region enclosed by γ and $|f(z)| > |g(z)|$ for all $z \in \gamma$. Then $f(z)$ and $f(z) + g(z)$ have the same numbers of zeros, counting multiplicities, in the region enclosed by γ.*

Proof. Without loss of generality, we assume that γ is counterclockwise oriented. Notice that $|f(z)| > |g(z)| \geq 0$ and $|f(z) + g(z)| \geq |f(z)| - |g(z)| > 0$ on γ. By the argument principle, it suffices to show that

$$\frac{1}{2\pi}\Delta_\gamma \arg f(z) = \frac{1}{2\pi}\Delta_\gamma \arg\,(f(z) + g(z)).$$

Moreover, we have,

$$f(z) + g(z) = f(z)\left(1 + \frac{g(z)}{f(z)}\right) \quad \text{for } z \in \gamma.$$

Therefore,

$$\begin{aligned} \Delta_\gamma \arg\,(f(z) + g(z)) &= \Delta_\gamma \arg\left[f(z)\left(1 + \frac{g(z)}{f(z)}\right)\right] \\ &= \Delta_\gamma \arg f(z) + \Delta_\gamma \arg\left(1 + \frac{g(z)}{f(z)}\right). \end{aligned} \tag{21.1}$$

Since

$$\left|\left(1 + \frac{g(z)}{f(z)}\right) - 1\right| = \frac{|g(z)|}{|f(z)|} < 1 \quad \text{for } z \in \gamma,$$

the image of γ under $1 + \dfrac{g(z)}{f(z)}$ is contained in $B_1(1)$. Hence, it does not enclose the origin. As a consequence,

$$\Delta_\gamma \arg\left(1 + \frac{g(z)}{f(z)}\right) = 0. \tag{21.2}$$

Combining (21.1) and (21.2), we complete the proof. □

Example 21.2. *In order to determine the number of roots, counting multiplicities, of the equation*

$$z^4 + 3z^3 + 6 = 0$$

inside the circle $\partial B_2(0)$, we let

$$f(z) = 3z^3 \quad \textit{and} \quad g(z) = z^4 + 6.$$

We have, on $\partial B_2(0)$,

$$|f(z)| = 24 \quad \textit{and} \quad |g(z)| \leq 16 + 6 = 22.$$

By Rouché's theorem, f and $f+g$ have the same numbers of roots, counting multiplicities, inside the circle. Since f has three roots, counting multiplicities, inside the circle, so does $f+g$. That is, the equation

$$z^4 + 3z^3 + 6 = 0$$

has three roots, counting multiplicities, inside the circle.

Example 21.3. *We can use Rouché's theorem to prove the fundamental theorem of algebra. Given a polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, \quad a_n \neq 0, \quad n \in \mathbb{N},$$

we want to show that P has n roots, counting multiplicities. Let

$$f(z) = a_n z^n \quad \text{and} \quad g(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1}.$$

For $R > 0$ sufficiently large, we have

$$|f(z)| > |g(z)| \quad \text{for } z \in \partial B_R(0).$$

By Rouché's theorem, both f and $f + g$ have n zeros, counting multiplicities, in $B_R(0)$. Hence, we conclude that $P(z) = f(z) + g(z)$ has n zeros, counting multiplicities, in $\mathbb{C}$.

22 Mappings by Elementary Functions

Linear Transformation. *A general non-constant linear transformation is defined by*

$$w = az + b$$

for some $a, b \in \mathbb{C}$, $a \neq 0$. Let $a = r_0 e^{i\theta_0}$. Recalling that, in Example 4.8, we have

(i) The mapping $w = z + b$ is a translation;

(ii) The mapping $w = e^{i\theta_0} z$ is a rotation;

(iii) The mapping $w = r_0 z$ is a scaling.

Therefore, the mapping $w = az + b$ can be regarded as a rotation and a scaling, followed by a translation. Moreover, it is one-to-one from $\mathbb{C}$ onto $\mathbb{C}$.

Example 22.1. *We consider the mapping*

$$w = (1 + i)z + 2.$$

Since

$$1 + i = \sqrt{2} e^{i\pi/4},$$

this mapping is the rotation counterclockwise with angle $\pi/4$ and expansion by factor $\sqrt{2}$, followed by a translation $z \mapsto z + 2$. Thus, it transforms the rectangle region with vertices 0, 1, $1 + 2i$ and $2i$ into another rectangle region with vertices 2, $3 + i$, $1 + 3i$ and $2i$.

Mapping by 1/z. *We consider the mapping*

$$w = \frac{1}{z},$$

which establishes a one-to-one correspondence between $\mathbb{C}\backslash\{0\}$ and $\mathbb{C}\backslash\{0\}$. We have

$$w = \frac{\overline{z}}{|z|^2}.$$

Hence,

$$|w| = \frac{1}{|z|} \quad \text{and} \quad \arg w = \arg \overline{z}.$$

Also, we have

$$|\overline{w}| = \frac{1}{|z|} \quad \text{and} \quad \arg \overline{w} = \arg z.$$

Therefore, it maps points interior to the unit circle to the exterior radially and vice versa, and maps points on the unit circle to itself. And then it is followed by a reflection with respect to the real axis.

In the following, we show that the mapping $w = 1/z$ transforms circles and lines into circles and lines. If

$$w = u + vi = \frac{1}{z} = \frac{1}{x + yi},$$

then

$$u = \frac{x}{x^2 + y^2} = \frac{x}{|z|^2} \quad \text{and} \quad v = -\frac{y}{x^2 + y^2} = -\frac{y}{|z|^2}. \tag{22.1}$$

Conversely, if

$$z = x + yi = \frac{1}{w} = \frac{1}{u + vi},$$

then

$$x = \frac{u}{|w|^2} \quad \text{and} \quad y = -\frac{v}{|w|^2}.$$

For a given circle, it can be described by the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

with $R > 0$. And for a given line, it can be described by the equation

$$ax + by = c$$

with a and b are not both zero. Now, we consider the equation

$$A\left(x^2 + y^2\right) + Bx + Cy + D = 0, \tag{22.2}$$

where A, B, C and D are real numbers with $B^2 + C^2 > 4AD$. Such an equation represents an arbitrary circle or line. When $A \neq 0$, (22.2) can be rewritten as

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2.$$

When $A = 0$, (22.2) becomes

$$Bx + Cy + D = 0$$

with $B^2 + C^2 > 0$, that is, B and C are not both zero. Putting (22.1) into (22.2) gives

$$A\left(u^2 + v^2\right)|z|^4 + Bu|z|^2 - Cv|z|^2 + D = 0.$$

Then, using the relation,

$$(u^2 + v^2)(x^2 + y^2) = |w|^2|z|^2 = 1,$$

we conclude that u and v satisfy

$$D\left(u^2 + v^2\right) + Bu - Cv + A = 0.$$

To summarize, under the mapping $w = 1/z$,

(i) *a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) is mapped onto a circle not passing through the origin;*

(ii) *points on a circle ($A \neq 0$) through the origin ($D = 0$) are mapped onto a line not passing through the origin;*

(iii) *a line ($A = 0$) not passing through the origin ($D \neq 0$) is mapped onto a circle through the origin except for the origin;*

(iv) *points on a line ($A = 0$) through the origin ($D = 0$) are mapped onto a line through the origin except for the origin.*

Remark 22.2. *By introducing the extended complex numbers $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the mapping $w = 1/z$ is one-to-one from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. w maps 0 and ∞ to ∞ and 0, respectively.*

Example 22.3. *According to the above derivation, a vertical line $x = c_1$ with $c_1 \neq 0$ is transformed by the mapping $w = 1/z$ onto the circle*

$$-c_1\left(u^2 + v^2\right) + u = 0,$$

or

$$\left(u - \frac{1}{2c_1}\right)^2 + v^2 = \left(\frac{1}{2c_1}\right)^2,$$

except for the origin. A horizontal line $y = c_2$ with $c_2 \neq 0$ is transformed by the mapping $w = 1/z$ onto the circle

$$c_2\left(u^2 + v^2\right) + v = 0,$$

or

$$u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \left(\frac{1}{2c_2}\right)^2,$$

except for the origin.

Example 22.4. *Under the mapping $w = 1/z$, we show that the half plane $\{z : \operatorname{Re} z > c_1\}$ with $c_1 > 0$ is mapped onto the disc $\{w : |w - 1/2c_1| < 1/2c_1\}$. For any $c > c_1$, by Example 22.3, the line $\{z : \operatorname{Re} z = c\}$ is mapped onto $\{w \neq 0 : |w - 1/2c| = 1/2c\}$. Since*

$$\{z : \operatorname{Re} z > c_1\} = \bigcup_{c>c_1} \{z : \operatorname{Re} z = c\}$$

and

$$\left\{w : \left|w - \frac{1}{2c_1}\right| < \frac{1}{2c_1}\right\} = \bigcup_{c>c_1} \left\{w \neq 0 : \left|w - \frac{1}{2c}\right| = \frac{1}{2c}\right\},$$

we conclude that half plane $\{z : \operatorname{Re} z > c_1\}$ is mapped onto the disc $\{w : |w - 1/2c_1| < 1/2c_1\}$.

Linear Fractional Transformation. *A transformation*

$$w = \frac{az + b}{cz + d}, \tag{22.3}$$

where a, b, c and d are complex numbers with $ad - bc \neq 0$, is called a linear fractional transformation or Möbius transformation. Since above definition can be written as

$$cwz + dw - az - b = 0, \tag{22.4}$$

it is also called a bilinear transformation.

If $c = 0$, the mapping (22.3) reduces to a linear transformation

$$w = \frac{a}{d}z + \frac{b}{d}, \quad ad \neq 0.$$

If $c \neq 0$, (22.3) can be rewritten as

$$w = \frac{a}{c} + \frac{bc - ad}{c} \cdot \frac{1}{cz + d}.$$

It can be regarded as a composition of mappings

$$Z = cz + d, \quad W = \frac{1}{Z}, \quad \text{and} \quad w = \frac{a}{c} + \frac{bc - ad}{c} W.$$

Such a mapping is one-to-one from $\mathbb{C}\backslash\{-d/c\}$ onto $\mathbb{C}\backslash\{a/c\}$. Moreover, it transforms circles and lines into circles and lines.

Remark 22.5. *The mapping $w = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$, is one-to-one from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. w maps $-d/c$ and ∞ to ∞ and a/c, respectively.*

Proposition 22.6. *Given three distinct points z_1, z_2, z_3 and three distinct points w_1, w_2, w_3, there is a unique linear fractional transformation that maps z_k to w_k, $k = 1, 2, 3$.*

Proof. Notice that a linear fractional transformation of the form (22.3) can be determined implicitly by (22.4). By putting (z_k, w_k) into (22.4), $k = 1, 2, 3$, we obtain three equations of four unknowns a, b, c and d. Thus, the ratios of the coefficients a, b, c and d can be uniquely determined. Consequently, the transformation is uniquely determined. $\square$

Example 22.7. *Find the linear fractional transformation*

$$w = \frac{az+b}{cz+d}$$

that maps points 2, i *and* -2 *to* 1, i *and* -1, *respectively. Taking the values of* w *at* $z=2$ *and* $z=-2$, *we have*

$$\frac{2a+b}{2c+d} = 1 \quad \textit{and} \quad \frac{-2a+b}{-2c+d} = -1.$$

The above equalities imply that

$$b = 2c \quad \textit{and} \quad d = 2a.$$

Thus,

$$w = \frac{az+2c}{cz+2a}.$$

Since i *is mapped to* i, *above equality gives* $c = ai/3$. *Therefore,*

$$w = \frac{az + \dfrac{2ai}{3}}{\dfrac{ai}{3}z + 2a} = \frac{3z+2i}{iz+6}.$$

Proposition 22.8. *The equation*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{22.5}$$

defines implicitly a linear fractional transformation that maps distinct points z_1, z_2, z_3 *to distinct points* w_1, w_2, w_3, *respectively.*

Proof. (22.5) can be written as

$$(w-w_1)(w_2-w_3)(z-z_3)(z_2-z_1) = (w-w_3)(w_2-w_1)(z-z_1)(z_2-z_3). \tag{22.6}$$

Putting $z = z_1$ into (22.6), we have

$$(w-w_1)(w_2-w_3)(z_1-z_3)(z_2-z_1) = 0.$$

It follows that $w = w_1$. Putting $z = z_2$ into (22.6), we have

$$(w-w_1)(w_2-w_3) = (w-w_3)(w_2-w_1)$$

It follows that $w = w_2$. Putting $z = z_3$ into (22.6), we have

$$0 = (w-w_3)(w_2-w_1)(z_3-z_1)(z_2-z_3).$$

It follows that $w = w_3$. Moreover, the mapping defined by (22.5) can be written in the form (22.4), i.e.,

$$cwz + dw - az - b = 0.$$

Since the mapping is not constant, the condition $ad - bc \neq 0$ is satisfied. Hence, (22.5) defines a linear fractional transformation mapping z_1, z_2 and z_3 to w_1, w_2 and w_3, respectively. $\square$

Example 22.9. *By using Proposition 22.8, the linear fractional transformation that maps points 2, i and -2 to 1, i and -1 we found in Example 22.7 can be obtained by solving the equation*

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}.$$

The above equation gives

$$w = \frac{3z+2i}{iz+6}.$$

Proposition 22.10. *For a linear fractional transformation, the following two statements are equivalents.*

(i) It maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the disc $\{w : |w| < 1\}$ and the boundary of the half plane $\{z : \operatorname{Im} z = 0\}$ into the boundary of the disc $\{w : |w| = 1\}$.

(ii) It has the form

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right),$$

where $\alpha \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 > 0$.

Remark 22.11. *In fact, the mapping maps $\{z : \operatorname{Im} z = 0\}$ onto $\{w : |w| = 1\}\backslash\{e^{i\alpha}\}$.*

Proof. Assuming (i), we let

$$w = \frac{az+b}{cz+d}. \tag{22.7}$$

Notice that the line $\{z : \operatorname{Im} z = 0\}$ is mapped to the circle $\{w : |w| = 1\}$. We have $|w| = 1$ if $z = 0$, which implies

$$|b| = |d| \neq 0.$$

Consider a sequence z_n on $\{z : \operatorname{Im} z = 0\}$ with $|z_n| \to \infty$, similarly, we have

$$|a| = |c| \neq 0.$$

We can rewrite (22.7) as

$$w = \frac{a}{c} \cdot \frac{z+b/a}{z+d/c}. \tag{22.8}$$

Since $|a/c| = 1$ and $|b/a| = |d/c| \neq 0$, (22.8) can be written as

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-z_1}\right) \tag{22.9}$$

for some $\alpha \in \mathbb{R}$ and $z_0, z_1 \in \mathbb{C}$ with $|z_0| = |z_1| \neq 0$. Now, putting $z = 1$ into (22.9), we have

$$|1-z_0| = |1-z_1|,$$

which implies

$$1 - z_0 - \overline{z_0} + |z_0|^2 = |1 - z_0|^2 = |1 - z_1|^2 = 1 - z_1 - \overline{z_1} + |z_1|^2.$$

Since $|z_0| = |z_1|$, we have

$$z_0 + \overline{z_0} = z_1 + \overline{z_1},$$

that is, $\operatorname{Re} z_0 = \operatorname{Re} z_1$. It follows that

$$z_1 = z_0 \quad \text{or} \quad z_1 = \overline{z_0}.$$

If $z_1 = z_0$, then w becomes a constant. Therefore, $z_1 = \overline{z_0}$, and we conclude that

$$w = e^{i\alpha} \left(\frac{z - z_0}{z - \overline{z_0}} \right).$$

Since the mapping maps z_0 to the origin, z_0 must be in the upper half plane $\{z : \operatorname{Im} z > 0\}$, that is, $\operatorname{Im} z_0 > 0$. Therefore, (ii) holds.

Conversely, assuming (ii), then the mapping is one-to-one from $\mathbb{C}\backslash\{\overline{z_0}\}$ onto $\mathbb{C}\backslash\{e^{i\alpha}\}$. Moreover, since

$$|w| = \frac{|z - z_0|}{|z - \overline{z_0}|},$$

we have

$$\begin{aligned} |w| < 1 \qquad & \text{if } \operatorname{Im} z > 0; \\ |w| = 1 \qquad & \text{if } \operatorname{Im} z = 0; \\ |w| > 1 \qquad & \text{if } \operatorname{Im} z < 0. \end{aligned}$$

Then (i) follows. □

Example 22.12. *The transformation*

$$w = \frac{i - z}{i + z}$$

can be written as

$$w = e^{i\pi} \left(\frac{z - i}{z - \overline{i}} \right).$$

Therefore, w maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the disc $\{w : |w| < 1\}$ and the line $\{z : \operatorname{Im} z = 0\}$ onto $\{w \neq -1 : |w| = 1\}$.

Example 22.13. *In this example, we will show that the transformation*

$$w = \frac{z - 1}{z + 1}$$

maps the half plane $\{z : \operatorname{Im} z > 0\}$ onto the plane $\{w : \operatorname{Im} w > 0\}$ and points on the real axis $\{z : \operatorname{Im} z = 0\}\backslash\{-1\}$ onto $\{w : \operatorname{Im} w = 0\}\backslash\{1\}$. We know that w is one-to-one from $\mathbb{C}\backslash\{-1\}$ onto $\mathbb{C}\backslash\{1\}$. Notice that if z is real, w is real. Since the image of the line $\{z : \operatorname{Im} z = 0\}$ must be on a line or a circle, it equals to $\{w : \operatorname{Im} w = 0\}\backslash\{1\}$. Moreover, by writing $z = x + yi$, we have

$$w = \frac{x - 1 + yi}{x + 1 + yi} = \frac{x^2 + y^2 - 1}{(x + 1)^2 + y^2} + \frac{2y}{(x + 1)^2 + y^2}\, i.$$

That is, $\operatorname{Im} w$ *has the same sign as* $\operatorname{Im} z$. *Therefore, w maps $\{z : \operatorname{Im} z > 0\}$ onto $\{w : \operatorname{Im} w > 0\}$.*

Example 22.14. *Let*

$$w = \log \frac{z-1}{z+1},$$

where the logarithm is the principal branch. w is one-to-one, and w is the composition of

$$Z = \frac{z-1}{z+1} \quad \textit{and} \quad w = \log Z.$$

Since Z maps the half plane $\{z : \operatorname{Im} z > 0\}$ onto the half plane $\{Z : \operatorname{Im} Z > 0\}$, the image of $\{z : \operatorname{Im} z > 0\}$ under $w = \log \dfrac{z-1}{z+1}$ is in the strip $\{w : 0 < \operatorname{Im} w < \pi\}$. Moreover, for each $0 < \theta_0 < \pi$, the mapping $w = \log Z$ maps each ray $\{Z : |Z| > 0, \operatorname{Arg} Z = \theta_0\}$ onto the line $\{w : \operatorname{Im} w = \theta_0\}$. We conclude that $w = \log \dfrac{z-1}{z+1}$ maps $\{z : \operatorname{Im} z > 0\}$ onto $\{w : 0 < \operatorname{Im} w < \pi\}$.

Mapping by $\mathbf{e^z}$. *Consider the mapping by the exponential function*

$$w = e^z.$$

w maps a vertical line $\{z : \operatorname{Re} z = c_1\}$, $c_1 \in \mathbb{R}$, onto the circle $\{w : |w| = e^{c_1}\}$. Each points on the circle is the image of infinitely many points. For a horizontal line $\{z : \operatorname{Im} z = c_2\}$, $c_2 \in \mathbb{R}$, w maps it one-to-one and onto the ray $\{w : |w| > 0,\, c_2 \in \arg w\}$. Vertical and horizontal line segments are mapped onto portions of circles and rays.

Example 22.15. *Let $w = e^z$. Then w maps the rectangular region*

$$\{z : a \le \operatorname{Re} z \le b,\, c \le \operatorname{Im} z \le d\}$$

onto the region

$$\left\{w : w = \rho e^{i\theta},\, e^a \le \rho \le e^b,\, c \le \theta \le d\right\}.$$

An vertical line segment

$$\{z : \operatorname{Re} z = c_1,\, c \le \operatorname{Im} z \le d\}, \quad a \le c_1 \le b,$$

is mapped onto the arc

$$\left\{w : w = e^{c_1} e^{i\theta},\, c \le \theta \le d\right\}.$$

An horizontal line segment

$$\{z : a \le \operatorname{Re} z \le b,\, \operatorname{Im} z = c_2\}$$

is mapped onto the line segment

$$\left\{w : w = \rho e^{i\theta},\, e^a \le \rho \le e^b,\, \theta = c_2\right\}.$$

Moreover, if $d - c < 2\pi$, the mapping one-to-one on the rectangular region.

Example 22.16. *Let $w = e^z$. Then w maps the strip*

$$\{z : 0 < \operatorname{Im} z < \pi\}$$

one-to-one and onto the half space

$$\{w : \operatorname{Im} w > 0\}.$$

Example 22.17. *We consider the mapping*

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

By letting $z = \rho e^{i\theta}$ and $w = u + vi$, we have

$$u = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\theta \quad \text{and} \quad v = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\theta.$$

For any positive $\rho \neq 1$, we have

$$\frac{u^2}{\left(\rho + \rho^{-1}\right)^2/4} + \frac{v^2}{\left(\rho - \rho^{-1}\right)^2/4} = 1.$$

For $\theta \in (-\pi, \pi) \setminus \{-\pi/2, 0, \pi/2\}$, we have

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1$$

Therefore,

(i) w maps a circle $\{z : |z| = \rho_0\}$ for positive $\rho_0 \neq 1$, onto an ellipse with foci ± 1 and length of long axis $\rho_0 + \rho_0^{-1}$;

(ii) w maps a ray $\{z : |z| \geq 1, \operatorname{Arg} z = \theta_0\}$ for $\theta_0 \notin \{-\pi/2, 0, \pi/2, \pi\}$ onto half a branch of hyperbola.

Example 22.18. *Consider the mapping*

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

As discussed in Example 22.17, the upper half circle $\{z : |z| = \rho, \operatorname{Im} z > 0\}$ for some $0 < \rho < 1$, is mapped onto

$$\left\{w = u + vi : \frac{u^2}{\left(\rho + \rho^{-1}\right)^2/4} + \frac{v^2}{\left(\rho - \rho^{-1}\right)^2/4} = 1,\, v < 0\right\},$$

the lower half part of an ellipse. The lower half circle $\{z : |z| = \rho, \operatorname{Im} z < 0\}$ for some $0 < \rho < 1$, is mapped onto

$$\left\{w = u + vi : \frac{u^2}{\left(\rho + \rho^{-1}\right)^2/4} + \frac{v^2}{\left(\rho - \rho^{-1}\right)^2/4} = 1,\, v > 0\right\},$$

the upper half part of an ellipse. As a consequence, w maps the upper half disc $\{z : |z| < 1, \operatorname{Im} z > 0\}$ onto the lower half plane $\{w : \operatorname{Im} w < 0\}$, and w maps the lower half disc $\{z : |z| < 1, \operatorname{Im} z < 0\}$ onto the upper half plane $\{w : \operatorname{Im} w > 0\}$.

Example 22.19. *Consider the mapping*

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

which can be regarded as a composition of

$$Z = e^{iz}, \quad W = iZ, \quad \text{and} \quad w = -\frac{1}{2}\left(W + \frac{1}{W}\right).$$

The half strip

$$\left\{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0\right\}$$

is mapped by $Z = e^{iz}$ onto the half disc

$$\left\{Z = \rho e^{i\theta} : 0 < \rho < 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}.$$

The above half disc is mapped by $W = iZ$ onto the half disc

$$\left\{W = \rho e^{i\theta} : 0 < \rho < 1, 0 < \theta < \pi\right\}.$$

Finally, the above is mapped by $w = -\frac{1}{2}\left(W + \frac{1}{W}\right)$ onto the half plane

$$\{w : \operatorname{Im} > 0\}.$$

That is, the half strip

$$\left\{z : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}, \operatorname{Im} z > 0\right\}$$

is mapped by $w = \sin z$ onto the half plane $\{w : \operatorname{Im} > 0\}$.

23 Conformal Mappings

Definition 23.1. *Let f be analytic at z_0. f is conformal at z_0 if there are $\theta_0 \in \mathbb{R}$ and $\rho_0 > 0$ such that for any smooth curve γ through z_0, f maps the tangent vector of γ at z_0 by rotating it by θ_0 and then scaling it by ρ_0. That is, for any γ through z_0, parametrized by $z(t)$, where $z(t_0) = z_0$, we have*

$$(f \circ z)'(t_0) = \rho_0 e^{i\theta_0} z'(t_0).$$

If f is conformal at each points on a region Ω, then the mapping by f is called a conformal mapping on Ω.

Proposition 23.2. *If f is analytic on a region Ω and $f'(z_0) \neq 0$ for some $z_0 \in \Omega$, then f is conformal at z_0.*

Proof. By writing $f'(z_0) = \rho_0 e^{i\theta_0}$ for some $theta_0 \in \mathbb{R}$ and $\rho_0 > 0$, then for any curve γ through z_0, parametrized by $z(t)$, where $z(t_0) = z_0$, we have

$$(f \circ z)'(t_0) = f'(z(t_0))z'(t_0) = f'(z_0)z'(t_0) = \rho_0 e^{i\theta_0} z'(t_0).$$

$\square$

Example 23.3. *The mapping $w = e^z$ is conformal on the whole complex plane $\mathbb{C}$.*